Lecture 19
April 27th, 2004

We give a slightly different proof of

**Theorem.** Let $\Omega$ a bounded domain in $\mathbb{R}^n$, and $1 \leq p < \infty$.

\[ W_0^{1,p}(\Omega) \subseteq C^{0,\alpha}(\Omega), \quad \alpha = 1 - \frac{n}{p}, \quad p > n, \]

and $\exists C(n,p, \Omega)$ such that for $u \in W_0^{1,p}(\Omega)$

\[ \|u\|_{C^{0,\alpha}(\Omega)} \leq C \cdot \|u\|_{W_0^{1,p}(\Omega)}, \quad \forall p > n, \]

in other words

\[ \sup_\Omega |u| + |u|_{C^{0,\alpha}(\Omega)} \leq C \cdot \left\{ \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \right\}, \quad \forall p > n. \]

Note the inequality is stronger than the one we stated in the previous lecture.

**Proof.** We take $u \in C_0^1(\Omega)$ as before, WLOG (density argument). Extend $u$ to $\mathbb{R}^n$ trivially, i.e set $u = 0$ on $\mathbb{R}^n \setminus \Omega$. Let $x, y \in \Omega$ and $\sigma = |x - y|$ and let $p$ be the point $\frac{x + y}{2}$. Put $B = B(p, \sigma)$ and take $z \in B$. By the Fundamental Theorem of Calculus

\[ u(x) - u(z) = \int_0^1 \frac{d}{dt} u(x + (1 - t)z) dt \]

\[ = \int_0^1 \nabla u(x + t(z - x)) \cdot (z - x) dt. \]

Integrating over $z \in B$
\[ |\int_B u(z)dz - \text{Vol}(B)u(x)| \leq \int_B \int_0^1 |\nabla u(x + t(z - x))| \cdot |z - x|dtdz \]
\[ \leq 2\sigma \int_B \int_0^1 |\nabla u(x + t(z - x))|dtdz \]
\[ = 2\sigma \int_0^1 \left( \int_B |\nabla u(x + t(z - x))|dz \right)dt. \]

Change variables to
\[ \bar{z} := x + t(z - x), \quad \rightarrow \quad d\bar{z} = t^n dz. \]

For \( z \in B(x, \sigma) \Rightarrow \bar{z} \in B(x, t\sigma) =: \bar{B}. \) In the new variable we have now
\[ |\int_B udz - \text{Vol}(B)u(x)| \leq 2\sigma \int_0^1 t^{-n} \left( \int_{\bar{B}} |\nabla u(\bar{z})|d\bar{z} \right)dt. \]

By the H"older Inequality for \( q \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \)
\[ \int_{\bar{B}} |\nabla u(\bar{z})|d\bar{z} \leq \left\{ \int_{\bar{B}} 1^q \right\}^{\frac{1}{q}} \cdot \left\{ \int_{\bar{B}} |\nabla u(w)|^p dw \right\}^{\frac{1}{p}} \]
\[ = \text{Vol}(B(t\sigma))^\frac{1}{q} ||\nabla u||_{L^p(\bar{B})} \]
\[ \leq \text{Vol}(B(t\sigma))^\frac{1}{q} ||\nabla u||_{L^p(\Omega)} \]
\[ = \omega_n^\frac{1}{q} t^\frac{n}{p} \sigma^\frac{n}{q} ||\nabla u||_{L^p(\Omega)} \quad \Rightarrow \]
\[ |\int_B udz - \text{Vol}(B)u(x)| \leq 2\sigma^{1 + \frac{n}{q}} \omega_n^{\frac{1}{q}} \left( \int_0^1 t^{-n} \cdot t^\frac{n}{p} dt \right) ||\nabla u||_{L^p(\Omega)}. \]

Divide now throughout by \( \text{Vol}(B) = \omega_n \sigma^n \)
\[ |\int_B u(z)dz - u(x)| \leq \sigma^{1 + \frac{n}{q} - n} \omega_n^{\frac{1}{q} - 1} \left( \int_0^1 t^{-n(1 - \frac{1}{p})} dt \right) ||\nabla u||_{L^p(\Omega)} \]
\[ = \sigma^{1 - \frac{n}{p} + \frac{n}{q}} \omega_n^{\frac{1}{q}} \left( \int_0^1 t^{-n(\frac{1}{p})} dt \right) ||\nabla u||_{L^p(\Omega)}. \]
and the integral evaluates to \[ \left. \frac{t^{\frac{n}{p} + 1}}{1 - \frac{n}{p}} \right|_0 \] which is finite iff \( p > n \). We thus conclude

\[
| \int_B u(z)dz - u(x) | \leq c(n, p) \cdot \sigma^{1 - \frac{n}{p}} \| \nabla u \|_{L^p(\Omega)}. 
\]

We repeat the above computation with \( x \) replaced by \( y \) and use the triangle inequality, which gives us

\[
| u(x) - u(y) | \leq 2c(n, p) \cdot | x - y |^{1 - \frac{n}{p}} \| \nabla u \|_{L^p(\Omega)}
\]

and subsequently

\[
\frac{| u(x) - u(y) |}{| x - y |^{1 - \frac{n}{p}}} \leq 2c(n, p) \cdot \| \nabla u \|_{L^p(\Omega)}.
\]

And concluding

\[
\| u \|_{C^\infty(\Omega)} = \| u \|_{L^\infty(\Omega)} + \sup_{x \neq y \in \Omega} \frac{| u(x) - u(y) |}{| x - y |^{1 - \frac{n}{p}}} \leq C(n, p, \Omega) \cdot \| \nabla u \|_{L^p(\Omega)}. 
\]

since both \( C^0 \) and \( L^\infty \) norms coincide, being just \( \sup_{\Omega} \), and finally because by our above computations we can also bound the \( L^\infty \) norm in terms of the \( L^p \) norm of \( Du \)

\[
| u(x) | \leq 2c(n, p, \text{diam}(\Omega)) \cdot \| \nabla u \|_{L^p(\Omega)}
\]

so \( \| u \|_{L^\infty(\Omega)} \) is bounded by the same \( \text{RHS} \).

\[ \blacksquare \]

**Compactness Theorems**

**Lemma.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), and \( 1 \leq p < \infty \). Let \( S \) be a bounded set in \( L^p(\Omega) \). In other words,

\[
\forall u \in S, \quad \| u \|_{L^p(\Omega)} \leq M_S.
\]
Suppose \( \forall \epsilon > 0, \exists \delta > 0 \) such that

\[
\forall \ u \in S, \ \forall \ |z| < \delta \ \int_{\Omega} |u(y+z) - u(y)|^p dy < \epsilon.
\]

Then \( S \) is precompact in \( L^p(\Omega) \) (denoted \( S \subset L^p(\Omega) \)), i.e every sequence of functions in \( S \) has convergent subsequence ("subconverges"), or equivalently \( \bar{S} \) is compact.

This is an Arzelà-Ascoli type theorem: bounded equicontinuous family is precompact. We just have to show somehow that the integral equicontinuity-type condition implies equicontinuity.

**Proof.** Mollify \( u \) as done previously in the course

\[
u_h = \int_{\mathbb{R}^n} \rho_h(x-y)u(y)dy, \quad \rho_h(z) = \frac{1}{h^n} \rho\left(\frac{|z|}{h}\right).
\]

Set \( S_h := \{u_h, u \in S\} \).

We compute

\[
u_h = \int_{\mathbb{R}^n} \rho_h(x-y)u(y)dy = \int_{\mathbb{R}^n} \rho_h(x-y)|u(y)|dy
\]

\[
= \int_{\mathbb{R}^n} \rho_h^\frac{1}{p} \rho_h^\frac{1}{q} |u(y)|dy
\]

\[
\leq \left\{ \int_{\mathbb{R}^n} \rho_h \right\}^{\frac{1}{q}} \cdot \left\{ \rho_h |u(y)|^\frac{1}{p} \right\} dy
\]

\[
\leq ||u||_{L^p(\Omega)}.
\]

Now

\[
u_h(x+z) - \nu_h(x) = \int_{\mathbb{R}^n} \left[ \rho_h(x+z-y) - \rho_h(x-y) \right] u(y)dy
\]

\[
= \int_{\mathbb{R}^n} \left[ \rho_h(x-y) \left[ u(y+z) - u(y) \right] \right] dy
\]

and the same estimate as above yields

\[
u_h(x+z) - \nu_h(x) \leq 1 \cdot \left\{ \int_{\Omega} |u(y+z) - u(y)|^p \right\}^{\frac{1}{p}} \leq \epsilon^{\frac{1}{p}}.
\]
Now by our assumption for $\delta > 0$ small enough and $|z| < \delta$ we will attain any desired $\epsilon$ on the rhs. Note $\int_{\mathbb{R}^n} \rho_h = 1$ is fixed for all $h$ by our choice of $\rho$. Hence by definition we see that $S_h$ is an equicontinuous family, and bounded wrt the $L^p(\Omega)$ norm as inside $S$, hence by the Arzelà-Ascoli theorem $S_h$ is precompact in the space $L^p(\Omega)$.

Now $\lim_{h \to 0} S_h \to S$ as we have seen in previous lectures. So as the above estimates are independent of $h$, $S$ is precompact itself in $L^p(\Omega)$.

\textbf{Theorem (Kodrachov)} Let $\Omega$ be bounded in $\mathbb{R}^n$.

\begin{align*}
(I) & \quad p < n : \quad W_0^{1,p}(\Omega) \subseteq L^q(\Omega) \quad \forall \ 1 \leq q < \frac{np}{n-p}, \\
(II) & \quad p > n : \quad W_0^{1,p}(\Omega) \subseteq C^{0,\alpha}(\overline{\Omega}) \quad \forall \ 0 < \alpha < 1 - \frac{n}{p}
\end{align*}

and moreover $W_0^{1,p}(\Omega)$ is compactly embedded in each of the rhs.

We have then a curious situation— $W_0^{1,p}(\Omega) \subseteq L^{\frac{np}{n-p}} \subseteq L^q$ for $1 \leq q < \frac{np}{n-p}$ but the first inclusion is only continuous! Only for $q$ strictly smaller than $\frac{np}{n-p}$ is it compact... And similarly for the case $p > n$.

For the sake of clarity: we say $B_1 \subseteq B_2$ is compactly embedded if for every bounded set $S$ in $B_1$, $i(S) \subseteq B_2$ is precompact, where $i : B_1 \to B_2$ is the inclusion map.

\textbf{Proof. Case $q = 1$.} By the density argument we mentioned repeatedly we assume \textit{wlog} $S \subseteq C_0^1(\Omega)$ and that $M_S = 1$. Let $u \in S$. Then $||u||_{L^p(\Omega)} \leq 1$, $||Du||_{L^p(\Omega)} \leq 1$. Hence $||u||_{L^1(\Omega)} = \int_{\Omega} |u(x)| \leq \left\{ \int_{\Omega} 1 \right\}^{\frac{1}{p}} \left\{ \int_{\Omega} |u|^p \right\}^{\frac{1}{p}} \leq \text{Vol}(\Omega) \cdot 1$, in other words $S$ is also bounded in $L^1$. Once we show the condition of the Lemma holds then we will have precompactness in $L^1(\Omega)$. And indeed

\[ u(y + z) - u(y) = \int_0^1 \frac{du}{dt}(y + tz)dt = \int_0^1 \nabla u(y + tz) \cdot zdt \Rightarrow \]

\[ \int_{\Omega} |u(y + z) - u(y)|dy \leq |z|\text{Vol}(\Omega) \frac{1}{p} \||\nabla u||_{L^p(\Omega)} \leq c|z|. \]
Case $1 < q < \frac{np}{n-p}$. We try to find some estimates for the $L^q(\Omega)$ norm using the indispensible Hölder Inequality. Naturally we will be able to take care of boundedness of all such $q$ together if we allude to the fact that the $\Lambda^{\frac{np}{n-p}}(\Omega)$ is bounded, indeed the $L^p$ norms are increasing in $p$– first choose $\lambda$ such that 
\[ q\lambda + q(1 - \lambda)\frac{n-p}{np} = 1 \]
\[
\int |u|^q = \int |u|^{q\lambda} \cdot |u|^{q(1 - \lambda)} \leq \left\{ \int (|u|^{q\lambda})^{\frac{np}{n-p}} \cdot \frac{1}{q(1 - \lambda)} \right\}^{q(1 - \lambda)}(\frac{np}{n-p}) \Rightarrow 
||u||_{L^q(\Omega)} \leq ||u||_{L^{\frac{np}{n-p}}(\Omega)}^{1 - \lambda} \cdot ||u||_{L^1(\Omega)}^{\lambda} \cdot c \cdot ||\nabla u||_{L^p(\Omega)}^{(1 - \lambda)} 
\leq ||u||_{L^1(\Omega)}^{\lambda} \cdot c \cdot 1 \leq c(n, p, \text{Vol}(\Omega)),
\]
where we applied our Theorem from the previous lecture. Now note that we are done using the $q = 1$ case: $S$ is bounded in $L^2(\Omega)$ and hence a subsequence converges in $L^q(\Omega)$, but then by the above inequality it will also converge in $L^2(\Omega)$!

Case $p > n$. By the Theorem of the previous lecture $W^{1,p}_0(\Omega) \subseteq C^{0,\alpha}(\overline{\Omega})$ continuously. But now $C^{0,\alpha}(\overline{\Omega}) \subseteq C^{0,\beta}(\overline{\Omega})$ compactly for any $0 \leq \beta < \alpha$ as mentioned in one of the previous lectures.

Remark. Replacing $W^{1,p}_0(\Omega)$ by $W^{1,p}(\Omega)$ (the completion of $C^1(\Omega)$ wrt the $W^{1,p}$ norm) in the above embedding theorems require that the domain be Lipschitz, i.e $\partial \Omega$ is of class $C^{0,1}$ (this is a local requirement).