Lecture 1

Mean Value Theorem

**Theorem 1** Suppose \( \Omega \subset \mathbb{R}^n \), \( u \in C^2(\Omega) \), \( \Delta u = 0 \) in \( \Omega \), and \( B = B(y, R) \subset \subset \Omega \), then

\[
u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u ds = \frac{1}{\omega_n R^n} \int_B u dx\]

**Proof:** By Green’s formula, for \( r \in (0, R) \), \( \int_{\partial B_r} \frac{\partial u}{\partial \nu} ds = \int_{B_r} \Delta u dx = 0 \). Thus

\[
0 = \int_{\partial B_r} \frac{\partial u}{\partial \nu} ds = \int_{\partial B_r} \frac{\partial u}{\partial r}(y + r\omega)ds = r^{n-1} \int_{S^{n-1}} \frac{\partial u}{\partial r}(y + r\omega)d\omega = r^{n-1} \frac{\partial}{\partial r} \left( r^{1-n} \int_{\partial B_r} u ds \right) = r^{n-1} \frac{\partial}{\partial r} \left( r^{1-n} \int_{\partial B_r} u ds \right)
\]

\[\Longrightarrow r^{n-1} \int_{\partial B_r} u ds = \text{const} \text{ for any } r.\]

But we also have

\[n\omega_n u_{min}(B_r) \leq \frac{1}{r^{n-1}} \int_{\partial B_r} u ds \leq n\omega_n u_{max}(B_r),\]

taking limit as \( r \to \infty \), we get for any \( r \)

\[u(y) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_r} u ds.\]

Integral it, we get the solid mean value thm. \( \blacksquare \)

**Remark 1** We have \( \Delta u \geq 0 \implies u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u ds \), and we call such \( u \) sub-harmonic, i.e. \( u \) lies below harmonic function sharing the same boundary values.

Also we have \( \Delta u \leq 0 \implies u(y) \geq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B} u ds \) and we call \( u \) super-harmonic.

Application: Maximum principle and uniqueness.
**Theorem 2** \( \Omega \subset \mathbb{R}^n, u \in C^2(\Omega), \Delta u \geq 0, \text{ if } \exists p \in \Omega \text{ s.t. } \)

\[
u(p) = \max_{\Omega} u
\]

then \( u \) is constant.

**Proof:** Let

\[
M = \sup_{\Omega} u, \quad \Omega_M = \{x \in \Omega | u(x) = M\}.
\]

\( \Omega_M \) is not empty because \( p \in M, \, \Omega_M \) is closed by continuity, \( \Omega_M \) is open by mean value inequality. Thus \( \Omega_M = M \), i.e. \( u \) is constant function.

**Corollary 1** \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}), \Delta u = 0, \text{ then if } \Omega \text{ bounded, we have } \inf_{\partial \Omega} u \leq \sup_{\partial \Omega} x \in \Omega. \)

**Corollary 2** \( u, v \in C^2(\Omega) \cap C^0(\overline{\Omega}), \Delta u = \Delta v \text{ in } \Omega, u = v \text{ on } \partial \Omega \implies u \equiv v \text{ on } \partial \Omega. \)

**Corollary 3** \( \Delta u \geq 0, \Delta v = 0, u \equiv v \text{ on } \partial \Omega \implies u \leq v \text{ in } \Omega. \) (Hence ”subharmonic”

**Application: Harnack Inequality.**

**Theorem 3** Suppose \( \Omega \) domain, \( u \in C^2(\Omega), \Delta u = 0, \Omega' \subset \subset \Omega, u \geq 0 \text{ in } \Omega, \text{ then } \exists \text{ constant } C = C(n, \Omega, \Omega') \text{ s.t. } \)

\[
\sup_{\Omega'} u \leq C \inf_{\Omega'} u.
\]

**Proof:** Let \( y \in \Omega', B(y, 4R) \subset \Omega. \) Take \( x_1, x_2 \in B(y, R) \), we have

\[
u(x_1) = \frac{1}{\omega_n R^n} \int_{B(x_1, R)} u \, dx \leq \frac{1}{\omega_n R^n} \int_{B(y, 2R)} u \, dx,
\]

\[
u(x_2) = \frac{1}{\omega_n (3R)^n} \int_{B(x_2, 3R)} u \, dx \geq \frac{1}{\omega_n (3R)^n} \int_{B(y, 2R)} u \, dx,
\]

\[
\implies u(x_1) \leq 3^n u(x_2),
\]

\[
\implies \sup_{B(y, R)} u(x_1) \leq 3^n \inf_{B(y, R)} u(x_2),
\]

Choose \( R \) little enough s.t. \( B(y, 4R) \subset \Omega \text{ for } \forall y \in \Omega'. \) Let \( x_1, x_2 \in \overline{\Omega} \text{ s.t. to be maximal and minimal point of } u \text{ in } \overline{\Omega'} \) respectively. We can cover \( \Omega' \) by \( N \) balls of radius \( R \) since \( \Omega' \) is compact, so we have

\[
\sup_{\Omega'} u \leq u(x_1) \leq 3^n u(x_1') \leq \cdots \leq 3^{nN} \inf_{\Omega'} u.
\]

This completes our proof.\( \blacksquare \)
Remark 2 1. A Harnack inequality implies $C^\alpha$ regularity for $0 < \alpha < 1$.
2. A positive (or more generally bounded above or below) harmonic function on $\mathbb{R}^n$ is constant.

A Priori Estimate for harmonic function.

**Theorem 4** $u \in C^\infty, \Delta u = 0, \Omega' \subset \Omega$. Then for multi-index $\alpha$, there exists constant $C = C(n, \alpha, \Omega, \Omega')$ s.t.
\[
\sup_{\Omega'} |D^\alpha u| \leq C \sup_{\Omega} |u|.
\]

**Proof:** Since $\frac{\partial}{\partial x_i} \Delta = \Delta \frac{\partial}{\partial x_i}$, $Du$ is also harmonic. So by mean value theorem and divergence theorems, we have for $B(y, R) \subset \Omega$,
\[
Du(y) = \frac{1}{\omega_n R^n} \int_{B(y, R)} Du \, dx = \frac{1}{\omega_n R^n} \int_{\partial B} u \nu \, ds
\]
\[
\implies |Du(y)| \leq \frac{n}{R} \sup_{\partial B} |u|
\]
\[
\implies |Du(y)| \leq \frac{n}{d(y, \partial \Omega)} \sup_{\Omega} |u|.
\]
By induction, we get the stated estimate for higher order derivatives. \[\blacksquare\]

**Remark 3** We can weaken the assumptions to $u \in C^2(\Omega)$: $u \in C^2(\Omega)$ and $\Delta u = 0 \implies u$ analytic. We will do this next time.

Green’s Representation Formula.

Suppose $\Omega$ is $C^1$ domain, $u, v \in C^2(\Omega)$.

Green’s 1\textsuperscript{st} identity:
\[
\int_{\Omega} v \Delta u \, dx + \int_{\Omega} Du \cdot Dv \, dx = \int_{\partial \Omega} v \frac{\partial u}{\partial \nu} \, ds.
\]

Green’s 2\textsuperscript{nd} identity:
\[
\int_{\Omega} (v \Delta u - u \Delta v) \, dx = \int_{\partial \Omega} \left( v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right) \, ds.
\]

Find solution for Laplacian:
\[
\Gamma(x) = \begin{cases} 
\frac{1}{n(2-n)\omega_n} |x|^{2-n}, & n > 2, \\
\frac{1}{2^n n} \log |x|, & n = 2.
\end{cases}
\]

Note that away from origin, $\Delta \Gamma(x) = 0$. 

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Theorem 5 Suppose $u \in C^2(\Omega)$, then for $y \in \Omega$, we have

$$u(y) = \int_{\partial \Omega} (u \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y) \frac{\partial u}{\partial \nu}) d\sigma + \int_{\Omega} \Gamma(x-y) \Delta u dx.$$ 

Proof: Take $\rho$ small enough s.t. $B_\rho = B_\rho(y) \subset \Omega$. Apply Green’s 2nd formula to $u$ and $v(x) = \Gamma(x-y)$, which is harmonic in $\Omega \setminus \{y\}$, on the domain $\Omega \setminus B_\rho$, we get

$$\int_{\Omega \setminus B_\rho} \Gamma(x-y) \Delta u dx = \int_{\partial \Omega} (\Gamma(x-y) \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu}(x-y)) d\sigma + \int_{\partial B_\rho, \nu} (\Gamma(x-y) \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu}(x-y))) d\sigma.$$ 

Let $\rho \to 0$, notice that as $\rho \to 0$

$$\left| \int_{\partial B_\rho} \Gamma(x-y) \frac{\partial u}{\partial \nu} d\sigma \right| \leq \Gamma(\rho) \sup_{\partial B_\rho} |Du| n \omega_n \rho^{n-1} \to 0,$$

$$\int_{\partial B_\rho, \nu} \frac{\partial \Gamma}{\partial \nu}(x-y) d\sigma = -\Gamma'(\rho) \int_{\partial B_\rho} u d\sigma = \frac{-1}{n \omega_n \rho^{n-1}} \int_{\partial B_\rho} u d\sigma \to -u(y),$$

thus we get the Green’s Representation Formula. 

Application of Green’s Formula:

Theorem 6 Let $B = B_R(0)$ and $\varphi$ is continuous function on $\partial B$. Then

$$u(x) = \begin{cases} \frac{R^2 - |x|^2}{n \omega_n R^2} \int_{\partial B} \frac{\varphi(y)}{|x-y|^n} d\sigma, & x \in B, \\ \varphi(x), & x \in \partial B. \end{cases}$$

belongs to $C^2(B) \cap C^0(\overline{B})$ and satisfies $\Delta u = 0$ in $B$. 

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