Outline

Conditional expectation

Martingales

Arcsin law, other SRW stories
Conditional expectation

Martingales

Arcsin law, other SRW stories
Conditional expectation

- Say we’re given a probability space \((\Omega, \mathcal{F}_0, P)\) and a \(\sigma\)-field \(\mathcal{F} \subset \mathcal{F}_0\) and a random variable \(X\) measurable w.r.t. \(\mathcal{F}_0\), with \(E|X| < \infty\). The **conditional expectation of \(X\) given \(\mathcal{F}\)** is a new random variable, which we can denote by \(Y = E(X|\mathcal{F})\).

- We require that \(Y\) is \(\mathcal{F}\) measurable and that for all \(A\) in \(\mathcal{F}\), we have \(\int_A XdP = \int_A YdP\).

- Any \(Y\) satisfying these properties is called a **version** of \(E(X|\mathcal{F})\).

- Is it possible that there exists more than one version of \(E(X|\mathcal{F})\) (which would mean that in some sense the conditional expectation is not canonically defined)?

- Is there some sense in which \(E(X|\mathcal{F})\) always exists and is always uniquely defined (maybe up to set of measure zero)?
Claim: Assuming $Y = E(X|\mathcal{F})$ as above, and $E|X| < \infty$, we have $E|Y| \leq E|X|$. In particular, $Y$ is integrable.

Proof: let $A = \{ Y > 0 \} \in \mathcal{F}$ and observe: 
\[
\int_A YdP \int_A XdP \leq \int_A |X|dP. \text{ By similarly argument, } \\
\int_{A^c} -YdP \leq \int_{A^c} |X|dP.
\]

Uniqueness of $Y$: Suppose $Y'$ is $\mathcal{F}$-measurable and satisfies 
$\int_A Y'dP = \int_A XdP = \int_A YdP$ for all $A \in \mathcal{F}$. Then consider the set $Y - Y' \geq \epsilon$. Integrating over that gives zero. Must hold for any $\epsilon$. Conclude that $Y = Y'$ almost everywhere.
Radon-Nikodym theorem

- Let $\mu$ and $\nu$ be $\sigma$-finite measures on $(\Omega, \mathcal{F})$. Say $\nu << \mu$ (or $\nu$ is **absolutely continuous w.r.t.** $\mu$) if $\mu(A) = 0$ implies $\nu(A) = 0$.

- Recall **Radon-Nikodym theorem**: If $\mu$ and $\nu$ are $\sigma$-finite measures on $(\Omega, \mathcal{F})$ and $\nu$ is absolutely continuous w.r.t. $\mu$, then there exists a measurable $f : \Omega \to [0, \infty)$ such that $\nu(A) = \int_A f d\mu$.

- Observe: this theorem implies existence of conditional expectation.
Outline

Conditional expectation

Martingales

Arcsin law, other SRW stories
Conditional expectation

Martingales

Arcsin law, other SRW stories
Two big results

- Optional stopping theorem: Can’t make money in expectation by timing sale of asset whose price is non-negative martingale.

- Martingale convergence: A non-negative martingale almost surely has a limit.
Wald’s equation: Let \( X_i \) be i.i.d. with \( E|X_i| < \infty \). If \( N \) is a stopping time with \( EN < \infty \) then \( ES_N = EX_1 EN \).

Wald’s second equation: Let \( X_i \) be i.i.d. with \( E|X_i| = 0 \) and \( EX_i^2 = \sigma^2 < \infty \). If \( N \) is a stopping time with \( EN < \infty \) then \( ES_N = \sigma^2 EN \).
Wald applications to SRW

- $S_0 = a \in \mathbb{Z}$ and at each time step $S_j$ independently changes by $\pm 1$ according to a fair coin toss. Fix $A \in \mathbb{Z}$ and let $N = \inf\{k : S_k \in \{0, A\}$. What is $\mathbb{E}S_N$?
- What is $\mathbb{E}N$?
Conditional expectation

Martingales

Arcsin law, other SRW stories
Outline

Conditional expectation

Martingales

Arcsin law, other SRW stories
Reflection principle

- How many walks from \((0, x)\) to \((n, y)\) that don’t cross the horizontal axis?
- Try counting walks that do cross by giving bijection to walks from \((0, -x)\) to \((n, y)\).
Ballot Theorem

Suppose that in election candidate $A$ gets $\alpha$ votes and $B$ gets $\beta < \alpha$ votes. What’s probability that $A$ is ahead throughout the counting?

Answer: $(\alpha - \beta)/(\alpha + \beta)$. Can be proved using reflection principle.
Theorem for last hitting time.
Theorem for amount of positive positive time.