18.175: Lecture 35

Ergodic theory

Scott Sheffield

MIT

Outline

Recall setup

Birkhoff's ergodic theorem

Outline

Recall setup

Birkhoff's ergodic theorem

Definitions

- Say that A is **invariant** if the symmetric difference between $\phi(A)$ and A has measure zero.
- Observe: class $\mathcal{I}$ of invariant events is a σ-field.
- Measure preserving transformation is called **ergodic** if $\mathcal{I}$ is trivial, i.e., every set $A \in \mathcal{I}$ satisfies $P(A) \in \{0,1\}$.
- **Example:** If $\Omega = \mathbb{R}^{\{0,1,\ldots\}}$ and A is invariant, then A is necessarily in tail σ-field $\mathcal{T}$, hence has probability zero or one by Kolmogorov's $0-1$ law. So sequence is ergodic (the shift on sequence space $\mathbb{R}^{\{0,1,2,\ldots\}}$ is ergodic.
- **Other examples:** What about fair coin toss ($\Omega = \{H, T\}$) with $\phi(H) = T$ and $\phi(T) = H$? What about stationary Markov chain sequences?

Outline

Recall setup

Birkhoff's ergodic theorem

Outline

Recall setup

Birkhoff's ergodic theorem

Ergodic theorem

- Let ϕ be a measure preserving transformation of $(\Omega, \mathcal{F}, P)$. Then for any $X \in L^1$ we have

$$\frac{1}{n}\sum_{m=0}^{n-1} X(\phi^m \omega) \to E(X|\mathcal{I})$$

a.s. and in L^1.

- Note: if sequence is ergodic, then $E(X|\mathcal{I}) = E(X)$, so the limit is just the mean.
- Proof takes a couple of pages. Shall we work through it?
- There's this lemma: let A_k be the event the maximum M_k of X_0 and $X_0 + X_1$ up to $X_1 + \ldots + X_{k-1}$ is non-negative. Then $EX_0 1_{A_k} \geq 0$ is non-negative.

Benford's law

- Typical starting digit of a physical constant? Look up Benford's law.
- Does ergodic theorem kind of give a mathematical framework for this law?

18.175 Theory of Probability
Spring 2014