Ideal asymmetric junction elements

Relax the symmetry assumption and examine the resulting junction structure. For simplicity, consider two-port junction elements.

As before, assume instantaneous power transmission between the ports without storage or dissipation of energy. Characterize the power flow in and out of a two-port junction structure using four real-valued wave-scattering variables. Using vector notation:

\[
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (A.1)
\]

\[
\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (A.2)
\]

The input and output power flows are the square of the length of these vectors, their inner products.

\[
P_{\text{in}} = 2 \sum_{i=1}^{2} u_i^2 = \mathbf{u}^t \mathbf{u} \quad (A.3)
\]

\[
P_{\text{out}} = 2 \sum_{i=1}^{2} v_i^2 = \mathbf{v}^t \mathbf{v} \quad (A.4)
\]

The constitutive equations of the junction structure may be written as follows.

\[
\mathbf{v} = f(\mathbf{u}) \quad (A.5)
\]

Geometrically, the requirement that power in equal power out means that the length of the vector \( \mathbf{v} \) must equal the length of the vector \( \mathbf{u} \), i.e. their tips must lie on the perimeter of a circle (see figure A.1).

For any two particular values of \( \mathbf{u} \) and \( \mathbf{v} \), the algebraic relation \( f(\cdot) \) is equivalent to a rotation operator.

\[
\mathbf{v} = S(\mathbf{u}) \mathbf{u} \quad (A.6)
\]

where the square matrix \( S \) is known as a scattering matrix.
S need not be a constant matrix, but may in general depend on the power flux through the junction, hence the notation $S(u)$. However, $S$ is subject to important restrictions. In particular,

$$v^tv = u^tS^tSu = u^tu \quad (A.7)$$

$S$ is ortho-normal matrix: the vectors formed by each of its rows (or columns) are (i) orthogonal and (ii) have unit magnitude; its transpose is its inverse.

$$S^tS = 1 \quad (A.8)$$

This constrains the coefficients of the scattering matrix as follows.

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (A.9)$$

$$a^2 + c^2 = 1 \quad (A.10)$$

$$ab + cd = 0 \quad (A.11)$$

$$b^2 + d^2 = 1 \quad (A.12)$$
As there are only three independent equations and four unknown quantities, we see that this junction is characterized by a single parameter. We may also write the orthogonality condition as

\[ SS^t = 1 \]  \hspace{1cm} (A.13)

which yields the following equations.

\[ a^2 + b^2 = 1 \]  \hspace{1cm} (A.14)
\[ ac + bd = 0 \]  \hspace{1cm} (A.15)
\[ c^2 + d^2 = 1 \]  \hspace{1cm} (A.16)

There are four possible solutions to these equations. Combining A.10 and A.16, \[ a^2 = 1 - c^2 = d^2. \] Thus \[ a = \pm d. \]

If \[ a = d \] then \[ b = c = \pm \sqrt{1 - a^2}. \]

One solution

Choosing the positive root yields one solution. Assuming the coefficient \( a \) to be the undetermined parameter,

\[
S = \begin{bmatrix}
a & \sqrt{1 - a^2} \\
-\sqrt{1 - a^2} & a
\end{bmatrix}
\]  \hspace{1cm} (A.17)

Rewrite in terms of effort and flow variables.

\[
e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}
\]  \hspace{1cm} (A.18)
\[
f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}
\]  \hspace{1cm} (A.19)

The relation between efforts and wave-scattering variables is as follows.

\[
e = (u - v) c = c (1 - S) u
\]  \hspace{1cm} (A.20)

where \( c \) is a scaling constant. The relation between flows and wave-scattering variables is as follows.
\[ f = (u + v)/c = 1/c \ (1 + S) \ u \]  

(A.21)

If \(|a| \neq 1\) then \(1 + S\) and \(1 - S\) are nonsingular matrices and the input wave scattering variables \(u_1\) and \(u_2\) may be eliminated as follows.

\[
e = c^2 (1 - S) (1 + S)^{-1} f
\]

(A.22)

\[
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} = c^2 \begin{bmatrix}
0 & -\sqrt{(1 - a)/(1 + a)} \\
\sqrt{(1 - a)/(1 + a)} & 0
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
\]

(A.23)

Writing \(G = c^2 \sqrt{(1 - a)/(1 + a)}\) we obtain the equation for an ideal gyrator.

\[
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} = \begin{bmatrix}
0 & -G \\
G & 0
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
\]

(A.24)

Writing \(G = c^2 \sqrt{(1 - a)/(1 + a)}\) we obtain the equation for an ideal gyrator.

Note that equations A.20 and A.21 imply a sign convention in effort-flow coordinates such that power is positive inwards on both ports.

\[ P_{\text{net inwards}} = e^Tf = u^Tu - v^Tv \]  

(A.25)

To follow the more common sign convention we may simply change the sign of \(f_2\) in equation A.24.

If \(a = 1\), \(e\) is identically zero for all values of \(f\). No energy is exchanged between the ports and the junction structure behaves like a dissipator with zero resistance.

If \(a = -1\), \(f\) is identically zero for all values of \(e\). No energy is exchanged between the ports and the junction structure behaves like a dissipator with infinite resistance (zero conductance).
A second solution

Choosing \( a = d \) and using the negative root yields another solution. Again assuming the coefficient \( a \) to be the undetermined parameter,

\[
S = \begin{bmatrix}
a & -\sqrt{1 - a^2} \\
\sqrt{1 - a^2} & a
\end{bmatrix}
\]  \hspace{1cm} (A.26)

In this case the relation between efforts and flows is

\[
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} = c^2 \begin{bmatrix}
0 & \sqrt{(1 - a)/(1 + a)} \\
-\sqrt{(1 - a)/(1 + a)} & 0
\end{bmatrix} \begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
\]  \hspace{1cm} (A.27)

Again we obtain the equation for an ideal gyrator.

\[
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix} = \begin{bmatrix}
0 & G \\
-G & 0
\end{bmatrix} \begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
\]  \hspace{1cm} (A.28)
A third solution

If \( a = -d, b = c = \pm \sqrt{1 - a^2} \). Using the positive root and assuming \( a \) to be the undetermined parameter

\[
S = \begin{bmatrix}
a & \sqrt{1 - a^2} \\
\sqrt{1 - a^2} & -a
\end{bmatrix}
\]  \hspace{1cm} (A.29)

In this case the matrices \( 1 + S \) and \( 1 - S \) are singular for all values of the parameter \( a \).

However, equations A.20 and A.21 may be combined as follows:

\[
\begin{bmatrix}
e_1/c \\
e_2/c \\
cf_1 \\
cf_2
\end{bmatrix} = \begin{bmatrix}
1 - S \\
1 + S
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]  \hspace{1cm} (A.30)

\[
\begin{bmatrix}
e_1/c \\
e_2/c \\
cf_1 \\
cf_2
\end{bmatrix} = \begin{bmatrix}
1 - a & -\sqrt{1 - a^2} \\
-\sqrt{1 - a^2} & 1 + a \\
1 + a & \sqrt{1 - a^2} \\
\sqrt{1 - a^2} & 1 - a
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]  \hspace{1cm} (A.31)

If \( |a| \neq 1 \), the 4 x 2 matrix relating efforts and flows to the input scattering variables contains two nonsingular 2 x 2 submatrices.

\[
\begin{bmatrix}
e_2/c \\
cf_1
\end{bmatrix} = \begin{bmatrix}
-\sqrt{1 - a^2} & 1 + a \\
1 + a & \sqrt{1 - a^2}
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]  \hspace{1cm} (A.32)

\[
\begin{bmatrix}
e_1/c \\
cf_2
\end{bmatrix} = \begin{bmatrix}
1 - a & -\sqrt{1 - a^2} \\
\sqrt{1 - a^2} & 1 - a
\end{bmatrix} \begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\]  \hspace{1cm} (A.33)

Solving the second of these for \( u \) and substituting into the first we obtain a relation between efforts and flows.
\[
\begin{bmatrix}
e_2 \\
f_1
\end{bmatrix} = \begin{bmatrix}
-\sqrt{(1 + a)/(1 - a)} & 0 \\
0 & \sqrt{(1 + a)/(1 - a)}
\end{bmatrix} \begin{bmatrix}
e_1 \\
f_2
\end{bmatrix}
\] (A.34)

Writing \( T = \sqrt{(1 + a)/(1 - a)} \) we obtain the equation for an ideal transformer.

\[
\begin{bmatrix}
e_2 \\
f_1
\end{bmatrix} = \begin{bmatrix}
-T & 0 \\
0 & T
\end{bmatrix} \begin{bmatrix}
e_1 \\
f_2
\end{bmatrix}
\] (A.35)

To follow the more common sign convention we may change the sign of \( e_2 \).

If the parameter \( a = \pm 1 \), an argument similar to that used above shows that a degenerate case results in which no energy is exchanged between the ports.
Final solution

Choosing \( a = d \) and using the negative root we obtain the fourth solution.

\[
S = \begin{bmatrix}
a & -\sqrt{1 - a^2} \\
-\sqrt{1 - a^2} & -a
\end{bmatrix}
\]  \hspace{1cm} (A.36)

Once again, the matrices \( 1 + S \) and \( 1 - S \) are singular for all values of the parameter \( a \), but by rearranging equations A.20 and A.21 as before the corresponding relation between efforts and flows is

\[
\begin{bmatrix}
e_2 \\
f_1
\end{bmatrix} = \begin{bmatrix}
\sqrt{(1 + a)/(1 - a)} & 0 \\
0 & -\sqrt{(1 + a)/(1 - a)}
\end{bmatrix} \begin{bmatrix}
e_1 \\
f_2
\end{bmatrix}
\]  \hspace{1cm} (A.37)

Again we obtain the equation for an ideal transformer

\[
\begin{bmatrix}
e_2 \\
f_1
\end{bmatrix} = \begin{bmatrix}
T & 0 \\
0 & -T
\end{bmatrix} \begin{bmatrix}
e_1 \\
f_2
\end{bmatrix}
\]  \hspace{1cm} (A.38)
Two-port junction elements

There are only two possible power-continuous, asymmetric two-port junction elements, the gyrator and the transformer.

Unlike the ideal symmetric junction elements (0 and 1) the ideal asymmetric junction elements may be nonlinear.

The relation between efforts and flows must have a *multiplicative* form.

The general asymmetric junction elements are a modulated gyrator (MGY) and a modulated transformer (MTF) respectively.