Problem 7.03

This problem is from “Advanced Fluid Mechanics Problems” by A.H. Shapiro and A.A. Sonin

A metal ball falls at steady speed in a large tank containing a viscous liquid. The ball falls so slowly that it is known that the inertia forces may be ignored in the equation of motion compared with the viscous forces.

(a) Perform a dimensional analysis of this problem, with the aim of relating the speed of fall $V$, to the diameter of the ball $D$, the mass density of the ball $\rho_b$, the mass density of the liquid $\rho_l$, and any other variables which play a role. Note that the “effective weight” of the ball is proportional to $(\rho_b - \rho_l)g$.

(b) Suppose that an iron ball (sp. gr. = 7.9, $D = 0.3$ cm) falls through a certain viscous liquid (sp. gr. = 1.5) at a certain steady-state speed. What would be the diameter of an aluminum ball (sp. gr. = 2.7) which would fall through the same liquid at the same speed assuming inertial forces are negligible in both flows?
Solution:

(a) Non-dimensional Groups

In steady state, the body force (weight, $W$) must be balanced with buoyancy ($F_B$) and drag ($F_D$) forces.

$$W_{eff} = (\rho_b - \rho_l)g \left( \frac{4}{3} \pi \left( \frac{D}{2} \right)^3 \right) = F_D$$  

(7.03a)

<table>
<thead>
<tr>
<th>$\rho_l$</th>
<th>$\mu$</th>
<th>$V$</th>
<th>$D$</th>
<th>$F_D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ML$^{-3}$]</td>
<td>[ML$^{-1}T^{-1}$]</td>
<td>[LT$^{-1}$]</td>
<td>[L$^1$]</td>
<td>[ML$^1T^{-2}$]</td>
</tr>
</tbody>
</table>

Thus we have

$n = 5$ variables

$k = 3$ primary variables

$\Rightarrow j = 2$ dimensionless group

For our primary variables, we choose (1) a fluid property: $\rho_l$, (2) a flow parameter: $V$, and (3) a geometric parameter: $D$. Therefore, the first dimensionless group is

$$\mu = f_1(\rho_l, V, D) \quad \text{or} \quad \Pi_1 = K_1 \rho_l^a V^b D^c$$

where $K_1$ is a constant. Thus,

$$M : 0 = 1 + a$$

$$L : 0 = -1 - 3a + b + c$$

$$T : 0 = -1 - b$$

$\Rightarrow a = b = c = -1$

$$\Pi_1 = K_1 \frac{\mu}{\rho_l V D} = \frac{K_1}{Re}$$  

(7.03b)

Similarly, we can obtain the second non-dimensional parameter.

$$\Pi_2 = K_2 F_D \rho_l^a V^b D^c$$

$$\Rightarrow \Pi_2 = K_2 \frac{F_D}{\rho_l V^2 D^2}$$  

(7.03c)

When $K_2 = 2$, this becomes the drag coefficient $C_D$, i.e.,

$$C_D = \frac{F_D}{\frac{1}{2} \rho V^2 A}$$

where $A$ is a characteristic cross-section area.
(b) Example of Similarity

In part (a), we obtained two non-dimensional variables. In highly viscous flows or fast speed flows, the drag force is a function of the Reynolds number. However, if the speed of the ball is very small \((Re \ll 1)\), then the drag force is no longer a function of Reynolds number. When the non-dimensional parameters are consistent in two situations, the flow fields are also similar. Let’s make the drag coefficients are the same in the two cases.

\[
C_D = \frac{(\rho_i - \rho_l)g \left( \frac{4}{3} \pi \left( \frac{D_i}{2} \right)^3 \right)}{\rho V^2 D_i^2} = \frac{(\rho_a - \rho_l)g \left( \frac{4}{3} \pi \left( \frac{D_a}{2} \right)^3 \right)}{\rho V^2 D_a^2}
\]

where the subscripts \(i\) and \(a\) denote the iron and aluminum.

\[
\Rightarrow \frac{(7.9 - 1.5) \times (0.3)^3}{(0.3)^2} = \frac{(2.7 - 1.5) \times D_a^3}{D_a^2}
\]

Therefore, the diameter of an aluminum ball which satisfies the similarity is

\[
D_a = 1.6 \ cm \quad \text{(7.03d)}
\]