Lecture 18: Shortest Paths IV - Speeding up Dijkstra

Lecture Overview

- Single-source single-target Dijkstra
- Bidirectional search
- Goal directed search - potentials and landmarks

Readings


DIJKSTRA single-source, single-target

Initialize()
\[ Q \leftarrow V[G] \]
while \( Q \neq \emptyset \)
do \( u \leftarrow \text{EXTRACT\_MIN}(Q) \) (stop if \( u = t \! \)!) 
for each vertex \( v \in \text{Adj}[u] \)
do \( \text{RELAX}(u, v, w) \)

Observation: If only shortest path from \( s \) to \( t \) is required, stop when \( t \) is removed from \( Q \), i.e., when \( u = t \)
Bi-Directional Search

Note: Speedup techniques covered here do not change worst-case behavior, but reduce the number of visited vertices in practice.

Bi-D Search

Alternate forward search from $s$
- backward search from $t$
  (follow edges backward)
- $d_f(u)$ distances for forward search
- $d_b(u)$ distances for backward search

Algorithm terminates when some vertex $w$ has been processed, i.e., deleted from the queue of both searches, $Q_f$ and $Q_b$.

Subtlety: After search terminates, find node $x$ with minimum value of $d_f(x) + d_b(x)$. $x$ may not be the vertex $w$ that caused termination as in example to the left!

Find shortest path from $s$ to $x$ using $\Pi_f$ and shortest path backwards from $t$ to $x$ using $\Pi_b$. Note: $x$ will have been deleted from either $Q_f$ or $Q_b$ or both.
Minimum value for $d_f(x) + d_b(x)$ over all vertices that have been processed in at least one search (see Figure 3):

\[
\begin{align*}
    d_f(u) + d_b(u) &= 3 + 6 = 9 \\
    d_f(u') + d_b(u') &= 6 + 3 = 9 \\
    d_f(w) + d_b(w) &= 5 + 5 = 10
\end{align*}
\]
Goal-Directed Search or $A^*$

Modify edge weights with potential function over vertices.

$$w(u, v) = w(u, v) - \lambda(u) + \lambda(v)$$

Search toward target as shown in Figure 4:

Correctness

$$\bar{w}(p) = w(p) - \lambda_t(s) + \lambda_t(t)$$

So shortest paths are maintained in modified graph with $\bar{w}$ weights (see Figure 5).

To apply Dijkstra, we need $\bar{w}(u, v) \geq 0$ for all $(u, v)$.

Choose potential function appropriately, to be feasible.

Landmarks

Small set of landmarks $LCV$. For all $u \in V, l \in L$, pre-compute $\delta(u, l)$.

Potential $\lambda_{t}^{(l)}(u) = \delta(u, l) - \delta(t, l)$ for each $l$.

CLAIM: $\lambda_{t}^{(l)}$ is feasible.

Feasibility

$$\bar{w}(u, v) = w(u, v) - \lambda_{t}^{(l)}(u) + \lambda_{t}^{(l)}(v)$$

$$= w(u, v) - \delta(u, l) + \delta(t, l) + \delta(v, l) - \delta(t, l)$$

$$= w(u, v) - \delta(u, l) + \delta(v, l) \geq 0$$ by the $\Delta$-inequality

$$\lambda_{t}(u) = \max_{l \in L} \lambda_{t}^{(l)}(u) \text{ is also feasible}$$
Figure 3: Forward and Backward Search and Termination.
Figure 4: Targeted Search

Figure 5: Modifying Edge Weights.
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