LECTURE 17

• Readings: Start Section 5.2

Lecture outline

• Review of the Bernoulli process
• Definition of the Poisson process
• Basic properties of the Poisson process
  – Distribution of the number of arrivals
  – Distribution of the interarrival time
  – Distribution of the \( k^{\text{th}} \) arrival time
The Bernoulli Process: Review

1 2 \ldots

- Discrete time; success probability in each slot = $p$.
- PMF of number of arrivals in $n$ time slots: Binomial
- PMF of interarrival time: Geometric
- PMF of time to $k^{th}$ arrival: Pascal
- Memorylessness

- What about continuous arrival times?
  Example: arrival to a bank.
The Poisson Process: Definition

- Let $P(k, \tau) =$ Probability of $k$ arrivals in an interval of duration $\tau$.

- Assumptions:
  - Number of arrivals in disjoint time intervals are independent.
  - For VERY small $\delta$, we have:
    $$P(k, \delta) \approx \begin{cases} 
    1 - \lambda \delta & \text{if } k = 0 \\
    \lambda \delta & \text{if } k = 1 \\
    0 & \text{if } k > 0 
    \end{cases}$$
  - $\lambda =$ “arrival rate” of the process.
From Bernoulli to Poisson (1)

• Bernoulli: Arrival prob. in each time slot = \( p \)

• Poisson: Arrival probability in each \( \delta \) -interval = \( \lambda \delta \)

• Let \( n = t/\delta \) and \( p = \lambda \delta \):

\[
\text{Number of arrivals in a } t \text{-interval} = \text{Number of successes in } n \text{ time slots (Binomial)}
\]
From Bernoulli to Poisson (2)

\[ \begin{align*}
1 & \quad p = \lambda \delta \\
0 & \quad n \to \infty \\
\delta \to 0 & \quad t = n \delta
\end{align*} \]

- Number of arrivals in a \( t \)-interval as \( n \to \infty \) =

\[
\binom{n}{k} p^k (1 - p)^{n-k} = \binom{n}{k} \left( \frac{\lambda t}{n} \right)^k \left( 1 - \frac{\lambda t}{n} \right)^{n-k} \\
\overset{\text{Binomial}}{=} \frac{n!}{(n-k)! n^k} \left( \frac{\lambda t}{n} \right)^k \left( 1 - \frac{\lambda t}{n} \right)^n \left( 1 - \frac{\lambda t}{n} \right)^{-k} \quad \overset{\text{reorder terms}}{=} \left( \frac{\lambda t}{n} \right)^k \left( 1 - \frac{\lambda t}{n} \right)^{-k} \left( 1 \right) e^{-\lambda t} \overset{\text{Poisson}}{=} \frac{\lambda^k}{k!} e^{-\lambda t}
\]
PMF of Number of Arrivals

- $N$: number of arrivals in a $\tau$-interval, thus:

$$P(N = k) = P(k, \tau) = \frac{(\lambda \tau)^k e^{-\lambda \tau}}{k!} \quad \text{(Poisson)}$$

$$k = 0, 1, \ldots$$

- **Mean:** $E[N] = \lambda \tau$

- **Variance:** $\text{Var}(N) = \lambda \tau$

- **Transform:** $M_N(s) = e^{\lambda \tau(e^s-1)}$
Email Example

- You get email according to a Poisson process, at a rate of $\lambda = 0.4$ messages per hour. You check your email every thirty minutes.

- Prob. of no new messages $= \frac{(.2)^0 e^{-.2}}{0!} = e^{-0.2}$

- Prob. of one new message $= \frac{(.2)^1 e^{-.2}}{1!} = .2e^{-0.2}$
• $Y_1$: time of the 1\textsuperscript{st} arrival.

• "First order" interarrival time: 
  \[ f_{Y_1}(y) = \lambda e^{-\lambda y}, \quad y \geq 0 \] (Exponential)

• Why:
  \[ P(Y_1 \leq y) = 1 - P(0, y) = 1 - e^{-\lambda y} \]
Interarrival Time

0 \quad \begin{array}{c}
\begin{array}{c}
\text{time}
\end{array}
\end{array}
\quad y

- **Fresh Start Property**: The time of the next arrival is independent from the past.

- **Memoryless property**: Suppose we observe the process for $T$ seconds and no success occurred. Then the density of the remaining time for arrival is exponential.

- **Email Example**: You start checking your email. How long will you wait, in average, until you receive your next email? $E[Y_1] = \frac{1}{\lambda} = 2.5 \text{ hours}$
Time of $k^{th}$ Arrival

- $Y_k$ : time of the $k^{th}$ arrival.

- $T_k = Y_k - Y_{k-1} \quad k = 2, 3, \ldots$ : kth interarrival time

- It follows that:

$$Y_k = T_1 + T_2 + \ldots + T_k$$
Time of $k^{th}$ Arrival

- $Y_k$: time of the $k^{th}$ arrival.
- $f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \quad y \geq 0$ (Erlang) “of order $k$”
### Bernoulli vs. Poisson

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