Response to Discontinuous Input

We will continue looking at the constant coefficient first order linear DE
\[ \dot{y} + ky = q(t). \]

It has the integrating factors solution
\[ y = e^{-kt} \left( \int e^{kt} q(t) dt + c \right). \] (1)

In this note we want to do an example where the input \( q(t) \) is discontinuous.

The most basic discontinuous function is the **unit-step function** at a point \( a \), defined by:
\[ u_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a. \end{cases} \] (2)

(We leave its value at \( a \) undefined, though some books give it the value 0 there, others the value 1 there.)

**Example 1.** We’ll look again at Newton’s law of cooling and my root beer cooler:
\[ \dot{y} + ky = kf(t), \]
where, \( y(t) \) is the temperature inside the cooler and \( f(t) \) is the temperature of the air. It’s a nice, cool morning with constant temperature. Suddenly the sun comes out and the air warms up to a higher constant temperature. What’s the response of my cooler to this signal?

We’ll assume the sun comes out at time \( t = a \), my cooler starts at \( t = 0 \) with temperature 0 and (somewhat idealized) the air temperature jumps instantly from 0 to 20 at time \( t = a \). So \( f(t) = 20u_a(t) \) and our IVP is
\[ \dot{y} + ky = k20u_a(t), \quad y(0) = 0. \]

**Solution.** For \( t < a \) we have the input is 0. Since \( y(0) = 0 \), the response is \( y(t) = 0 \).

For \( t \geq a \) the DE becomes \( \dot{y} + ky = 20k \) with \( y(a) = 0 \). The solution (which we have found before) is \( y(t) = 20 + ce^{-kt} \). Now we use the initial condition \( y(a) = 0 \) to find the value of \( c \). We get \( c = -20e^{ka} \), so \( y(t) = 20 - 20e^{ka}e^{-kt} \) for \( t \geq a \).
We can now assemble the results for $t < a$ and $t \geq a$ into one expression; for the latter, we also put the exponent into a more suggestive form.

\[
\text{input } = 20u_a(t) \quad \rightarrow \quad \text{response } = y(t) = \begin{cases} 
0 & 0 < t < a; \\
20 - 20e^{-k(t-a)} & t \geq a.
\end{cases}
\]  

(3)

Note that the response is just the translation $a$ units to the right of the response to the unit-step input at 0.

Our next example continues the temperature model with a different discontinuous input. In this case, the physical input is an external bath which is initially ice-water at 0 degrees, then replaced by water held at a fixed temperature for a time interval, then replaced once more by ice-water. To model the input we need the \textbf{unit box function} on $[a, b]$

\[
u_{ab} = \begin{cases} 
1 & a \leq t \leq b \\
0 & 0 \leq a < b; \quad \text{otherwise}
\end{cases}
\]  

(4)

\textbf{Example 2.} Find the response of the system

\[
\dot{y} + ky = kq, \quad \text{with IC } y(0) = 0
\]

to input $q(t) = 20u_{ab}(t)$.

\textbf{Solution.} There are at least three ways to do this:

a) Express $u_{ab}$ as a sum of unit step functions and use (3) together with superposition of inputs;

b) Use the function $u_{ab}$ directly in a definite integral expression for the response;

c) Find the response in two steps: first use (3) to get the response $y(t)$ for the input $u_a(t)$; this will be valid up till the point $t = b$.

Then, to continue the response for values $t > b$, evaluate $y(b)$ and find the response for $t > b$ to the input 0, with initial condition $y(b)$.

We will follow (c), leaving the first two as exercises.

By (3), the response to the input $u_a(t)$ is:

\[
y(t) = \begin{cases} 
0 & 0 \leq t < a \\
20 - 20e^{-k(t-a)} & t \geq a.
\end{cases}
\]
This is valid up to $t = b$, since $u_{ab}(t) = u_a(t)$ for $t \leq b$. Evaluating at $b$,

$$y(b) = 20 - 20e^{-k(b-a)}. \quad (5)$$

For $t > b$ we have $u_{ab} = 0$, so the DE is just $\dot{y} + ky = 0$. This models exponential decay (our most important DE) and we know the solution:

$$y(t) = ce^{-kt}. \quad (6)$$

We determine $c$ from the initial value (5). Equating the initial values $y(b)$ from (5) and (6), we get:

$$ce^{-kb} = 20 - 20e^{-kb+ka}$$

from which:

$$c = 20e^{kb} - 20e^{ka}.$$ 

By (6):

$$y(t) = 20(e^{kb} - e^{ka})e^{-kt}, \quad t \geq b. \quad (7)$$

After combining exponents in (7) to give an alternative form for the response we assemble the parts, getting:

$$y(t) = \begin{cases} 
0 & 0 \leq t \leq a \\
20 - 20e^{-k(t-a)} & a < t < b \\
20e^{-k(t-b)} - 20e^{-k(t-a)} & t \geq b.
\end{cases} \quad (8)$$