18.03SC Practice Problems 35

Matrix exponentials

Solution suggestions

1. In this problem, $A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$ and we are interested in the equation $\dot{u} = Au$.

(a) Find a fundamental matrix $\Phi(t)$ for $A$.

The eigenvalues of $A$ are solutions to $(1 - \lambda)^2 + 4 = \lambda^2 - 2\lambda + 5 = 0$, which are in this case the complex conjugates $1 \pm 2i$.

Choose one of them, say $\lambda_1 = 1 + 2i$, and find an eigenvector $v_1$ for it.

The eigenvectors corresponding to $\lambda_1$ are the vectors in the killed by $A - \lambda_1 I = \begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix}$, so we can take, say, $v_1 = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$.

This means this equation has a complex-valued solution

$$u(t) = e^{\lambda_1 t} v_1 = e^{(1+2i)t} \begin{bmatrix} 1 \\ 2i \end{bmatrix} = e^t \begin{bmatrix} \cos 2t + i \sin 2t \\ 2i \cos 2t - 2 \sin 2t \end{bmatrix} = \begin{bmatrix} e^t \cos 2t \\ -2e^t \sin 2t \end{bmatrix} + i \begin{bmatrix} e^t \sin 2t \\ 2e^t \cos 2t \end{bmatrix}$$

By taking real and imaginary parts, we get two independent real solutions, and so can read off a fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^t \cos 2t & e^t \sin 2t \\ -2e^t \sin 2t & 2e^t \cos 2t \end{bmatrix}.$$

The fundamental matrix is not unique, and there are other correct solutions. However, you can check other solutions by observing that any other fundamental matrix $\Phi(t)$ should be related to this one by matching up initial conditions; i.e., it should satisfy the equation

$$\Phi(t) = \Phi(t)\Phi(0)^{-1}\Phi(0).$$

(b) Find the exponential matrix $e^{At}$.

The exponential matrix is the fundamental matrix uniquely determined by the identity initial conditions $e^{A0} = I$.

So we can compute the exponential matrix for this system from any fundamental matrix $\Phi(t)$ found in the first part by using the formula

$$e^{At} = \Phi(t)\Phi(0)^{-1}.$$
For us, \( \Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \), so \( \Phi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \), and

\[
e^{At} = \begin{bmatrix} e^t \cos 2t & e^t \sin 2t \\ -2e^t \sin 2t & 2e^t \cos 2t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} e^t \cos 2t & \frac{1}{2} e^t \sin 2t \\ -2e^t \sin 2t & e^t \cos 2t \end{bmatrix}.
\]

(c) Find the solution to \( \dot{u} = Au \) with \( u(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

The general strategy is to find the solution with a given initial condition directly from the exponential matrix:

\[
u(t) = e^{At} u(0) = \begin{bmatrix} e^t \cos 2t & \frac{1}{2} e^t \sin 2t \\ -2e^t \sin 2t & e^t \cos 2t \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t (\cos 2t + \sin 2t) \\ 2e^t (\cos 2t - \sin 2t) \end{bmatrix}.
\]

In our case, we could have also observed that this initial condition was the sum of the real and imaginary parts of the complex solution we found in part (a) and gotten the answer by adding the columns of our fundamental matrix \( \Phi(t) \).

(d) Find a solution to \( \dot{u} = Au + \begin{bmatrix} 5 \\ 10 \end{bmatrix} \). What is the general solution? What is the solution with \( u(0) = 0 \)?

We guess a constant solution

\[
u = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.
\]

Substituting this into the DE gives

\[
0 = A \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \end{bmatrix}.
\]

This implies

\[
u = -A^{-1} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 1 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.
\]

(You can check that this is indeed a solution by verifying that it satisfies the equation \( Au + \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 0 \).)

Since all homogeneous solutions are of the form \( e^{At} \begin{bmatrix} a \\ b \end{bmatrix} \), the general solution is then given by

\[
e^{At} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} e^t (a \cos 2t + \frac{b}{2} \sin 2t) + 1 \\ e^t (b \cos 2t - 2a \sin 2t) - 6 \end{bmatrix},
\]

for some constants \( a, b \).
To find the particular solution with \( \mathbf{u}(0) = \mathbf{0} \), we plug \( t = 0 \) into this expression, and get that
\[
\begin{bmatrix}
a + 1 \\
b - 6
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix},
\]
so the desired solution is given by the constants \( a = -1 \) and \( b = 6 \):
\[
\begin{bmatrix}
e^t (-\cos 2t + 3 \sin 2t) + 1 \\
e^t (6 \cos 2t + 2 \sin 2t) - 6
\end{bmatrix}.
\]

2. Suppose \( \mathbf{u}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) (the constant trajectory) and \( \mathbf{u}_2(t) = \begin{bmatrix} e^t \\ -e^t \end{bmatrix} \) are solutions to the equation \( \dot{\mathbf{u}} = \mathbf{Bu} \) for some matrix \( \mathbf{B} \).

(a) What is the general solution? What is the solution \( \mathbf{u}(t) \) with \( \mathbf{u}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \)? What is the solution with \( \mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)?

The general solution is
\[
\mathbf{u}(t) = a\mathbf{u}_1(t) + b\mathbf{u}_2(t) = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} e^t \\ -e^t \end{bmatrix}.
\]

We can find the particular solution that satisfies any given initial condition \( \mathbf{u}(0) \) explicitly by solving the system of equations given by \( a\mathbf{u}_1(0) + b\mathbf{u}_2(0) = \mathbf{u}(0) \) for \( a \) and \( b \).

Here \( \mathbf{u}_1(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \mathbf{u}_2(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \), so the solution with initial condition \( \mathbf{u}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \) is the constant trajectory
\[
2\mathbf{u}_1(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix},
\]
and the solution with initial condition \( \mathbf{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) is
\[
\frac{1}{2}\mathbf{u}_1(t) + \frac{1}{2}\mathbf{u}_2(t) = \begin{bmatrix} e^t/2 + 1/2 \\ -e^t/2 + 1/2 \end{bmatrix}.
\]

(b) Find a fundamental matrix, and compute the exponential \( e^{\mathbf{B}t} \). What is \( e^{\mathbf{B}} \)?

Since the two given solutions are linearly independent, a fundamental matrix is
\[
\begin{bmatrix}
1 & e^t \\
1 & -e^{-t}
\end{bmatrix}.
\]
Evaluating this matrix at $t = 0$ gives
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
which has inverse
$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix},$$
so the exponential matrix is
$$e^{Bt} = \begin{bmatrix} 1 & e^t \\ 1 & -e^t \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t + 1 & 1 - e^t \\ 1 - e^t & e^t + 1 \end{bmatrix}.$$ Evaluating $e^{Bt}$ at $t = 1$ gives us
$$e^B = \frac{1}{2}\begin{bmatrix} e + 1 & 1 - e \\ 1 - e & e + 1 \end{bmatrix}.$$ 

(c) **What are the eigenvalues and eigenvectors of $B$?**

The eigenvalues of $e^B$ are 1 and $e$, with eigenvectors
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
and
$$\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
respectively.

The eigenvalues of $B$ are the logarithms of the eigenvalues of $e^B$, so they are 0 and 1, with the same eigenvectors.

We could have also deduced this from the two solutions given in the problem statement.

(d) **What is $B$?**

We can find $B$ by using the answer to part (c).

From the equation for the first eigenvalue, $B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $B$ has the form
$$\begin{bmatrix} a & -a \\ c & -c \end{bmatrix}.$$ From the equation for the second eigenvalue,
$$\begin{bmatrix} a & -a \\ c & -c \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$
so $a = 1/2$ and $c = -1/2$. Thus,
$$B = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$