In Lecture 2 we mentioned that
\[ \|\partial^2 u\|_{C^0} \lesssim \|\Delta u\|_{C^0} \quad \text{and} \quad \|\partial^2 u\|_{C^1} \lesssim \|\Delta u\|_{C^1} \] (1)
for \( u \in C_c^3(\mathbb{R}^n) \). However, Korn showed that
\[ [\partial^2 u]_a \lesssim [\Delta u]_a \]
for any \( a \in (0, 1) \). We will spend the next few lectures proving results involving the \( \alpha \)-Hölder norms, culminating in a proof of the Schauder inequality. In the process we will also see why the bounds in (1) fail.

The \( \alpha \)-Hölder norms provide an intermediate measure of smoothness between \( C^0 \) and \( C^1 \), and offer valuable control on solutions to \( \Delta u = f \) and \( \Delta u = 0 \) with boundary conditions.

To approach Korn’s inequality, we will use an expression for \( \partial^2 u \) in terms of \( \Delta u \). This formula is derived from physical potential theory. The “gravitational” field in \( \mathbb{R}^n \) is given by
\[ F_n(x) = c_n \frac{x}{|x|^n} \]
for some constant \( c_n > 0 \) and \( x \neq 0 \). As we shall see, it is most convenient to choose \( c_n = \frac{1}{|S^{n-1}(1)|} \) to simplify divergence formulae for \( F_n \). In fact \( F_n \) is generated by the potential
\[ \Gamma_n(x) = \begin{cases} c'_n |x|^{-n+2} & \text{if } n \geq 3 \\ c'_2 \log |x| & \text{if } n = 2 \end{cases} \]
for appropriate constants \( c'_n > 0 \). That is, \( F_n = \nabla \Gamma_n \). It is easy to calculate that \( |\nabla \Gamma_n| \sim |x|^{-n+1} \) and \( |\nabla^2 \Gamma_n| \sim |x|^{-n} \) for large \( |x| \). Furthermore, \( \text{div} \, F_n = 0 \) at \( x \neq 0 \), so \( \Delta \Gamma_n = 0 \) for \( x \neq 0 \). We note in passing that \( \text{div} \, F_n = 0 \) may be derived without computation from the symmetry of \( F_n \) and the fact that
\[ \int_{S^{n-1}(r)} F_n \cdot \hat{n} = 1 \] (2)
is independent of \( r \).
For the remainder of the lecture, let $\Omega \subset \mathbb{R}^n$ be an open bounded region with smooth boundary $\partial \Omega$. From $\text{div} \ F_n = 0$ for $x \neq 0$ and (2), we may easily verify that

$$\int \begin{cases} 1 & \text{if } 0 \in \Omega \\ 0 & \text{if } 0 \notin \Omega \end{cases} F_n \cdot \hat{n} = (3) \int \Omega \Delta \Gamma_n = \delta_0.$$  

This reflects the physical (and distribution theoretic) interpretation that $\text{div} \ F_n = \Delta \Gamma_n = \delta_0$. We now verify that convolution against $\Gamma_n$ yields a solution to Poisson’s equation.

**Proposition 1.** If $f \in C^2_c(\mathbb{R}^n)$ and $u := \Gamma_n * f$, then $u \in C^2(\mathbb{R}^n)$ and $\Delta u = f$.

**Proof.** By definition,

$$u(x) = \int f(y) \Gamma_n(x - y) \, dy = \int \Gamma_n(y) f(x - y) \, dy.$$  

These integral expressions are well-defined because $f \in C^0_c(\mathbb{R}^n)$ and $\Gamma_n \in L^1_{\text{loc}}(\mathbb{R}^n)$. Also, standard dominated convergence arguments show that we may bring first derivatives under the integral sign:

$$\partial_j u = \int (\int \partial_j \Gamma_n(x - y) f(x - y) \, dy) \, dy.$$  

Again these expressions are well-defined because $f \in C^1$ and $\partial_j \Gamma_n \in L^1_{\text{loc}}(\mathbb{R}^n)$. Differentiating further, we have

$$\partial_i \partial_j u = \int \partial_i \partial_j \Gamma_n \partial_j f(x - y) \, dy.$$  

Note that we may not form a parallel expression with $\partial_i \partial_j \Gamma_n$, because $\partial_i \partial_j \Gamma_n \notin L^1_{\text{loc}}(\mathbb{R}^n)$. By continuity, it is sufficient to verify that

$$\int \Delta u = \int f$$  

for all regions $\Omega$ satisfying the previously stated conditions. By the divergence theorem,

$$\int \Delta u = \int_{\partial \Omega} \nabla u \cdot \hat{n} = \int \left( \int_{\mathbb{R}^n} f(y) \nabla \Gamma_n(x - y) \, dy \right) \cdot \hat{n} \, dA(x).$$  

We use Fubini to interchange the order of integration:

$$\int \Delta u = \int_{\mathbb{R}^n} f(y) \left( \int_{\Omega} \nabla \Gamma_n(x - y) \cdot \hat{n} \, dA(x) \right) \, dy.$$  

Now (3) implies that

$$\int \Delta u = \int_{\mathbb{R}^n} f(y) \chi_{\Omega}(y) \, dy = \int f.$$  

Having proven a solution to the equation $\Delta u = f$, the question of uniqueness naturally arises. Could other expressions for $u$ also solve Laplace’s equation? We establish uniqueness in the case that $u$ is compactly supported:

**Proposition 2.** If $u \in C^2_c(\mathbb{R}^n)$ and $f := \Delta u$, then $u = \Gamma_n * f$. 

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Proof. Let $w := \Gamma_n \ast f$, so $\Delta(u - w) = 0$. The maximum principle for harmonic functions shows that $\max_{B_R} |u - w| = \max_{\partial B_R} |u - w|$. Hence to verify that $u = w$ it is sufficient to show that

$$\lim_{R \to \infty} \max_{\partial B_R} |u - w| = 0.$$ 

Since $u$ is compactly supported, this is equivalent to showing that

$$\lim_{R \to \infty} \max_{\partial B_R} |w| = 0.$$ 

This is simple when $n \geq 3$. After all, if $\text{supp } u \subset B_{R_0}$ and $|x| = R \geq 2R_0$, we have

$$|w(x)| = \left| \int_{B_{R_0}} f(y)\Gamma_n(x - y) \, dy \right| \leq \|f\|_{L^1(\mathbb{R}^n)} \sup_{|z| \geq R/2} |\Gamma_n(z)| \to 0$$

as $R \to \infty$. This estimate fails when $n = 2$, because $\Gamma_2$ does not decay as $|x| \to \infty$. We therefore deploy a more careful analysis, relying on the fact that $f$ is the Laplacian of a compactly supported function. In particular,

$$\int_{\mathbb{R}^2} f(y) \, dy = \int_{B_{R_0}} \Delta u(y) \, dy = \int_{S_{R_0}} \nabla u(y) \cdot \hat{n} \, dA(y) = 0.$$ 

Hence when $|x| = R \geq 2R_0$,

$$|w(x)| = \left| \int_{B_{R_0}} f(y)\Gamma_2(x) \, dy + \int_{B_{R_0}} f(y)(\Gamma_2(x - y) - \Gamma_2(x)) \, dy \right|$$

$$= \left| \int_{B_{R_0}} f(y)|\Gamma_2(x - y) - \Gamma_2(x)| \, dy \right|$$

$$\leq \int_{B_{R_0}} |f(y)||y| \max_{|x, x-y|} |\nabla \Gamma_2| \, dy$$

$$\leq R_0 \|f\|_{L^1(\mathbb{R}^2)} \sup_{|z| \geq R/2} |\nabla \Gamma_2(z)| \to 0$$

as $R \to \infty$. 

Korn’s inequality bounds the regularity of $\partial^2 u$ in terms of $\Delta u$ for compactly supported functions. We therefore wish to adapt the expressions in Proposition 1 to derive formulæ for the second partials of $u$. However, as noted in the proof of Proposition 1, this goal is complicated by the fact that $\partial_i \partial_j \Gamma_n \notin L^1_{\text{loc}}(\mathbb{R}^n)$. Hence we may not directly write $\partial_i \partial_j u = (\Delta u) \ast \partial_i \partial_j \Gamma_n$. We might hope that the integral defining the convolution converges conditionally, i.e. that

$$\partial_i \partial_j u = \lim_{\epsilon \to 0^+} \int_{|x-y| > \epsilon} f(y)\partial_i \partial_j \Gamma_n(x - y) \, dy$$

(4)

for $u \in C^4_c(\mathbb{R}^n)$. However, this equation is patently false if we recall that $\Delta \Gamma_n(z) = 0$ for $z \neq 0$. If we use (4) with $i = j$ and sum over $1 \leq i \leq n$, we find $\Delta u = 0$, regardless of the choice of $u$. Hence we need to account somehow for the effect of the singularity of $\Gamma_n$ on derivatives of the convolution $(\Delta u) \ast \Gamma_n$. As it turns out,
(4) is almost correct:

**Proposition 3.** If \( f \in C^2_c(\mathbb{R}^n) \) and \( u := f * \Gamma_n \), then

\[
\partial_i \partial_j u(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y| > \varepsilon} f(y) \partial_i \partial_j \Gamma_n(x-y) \, dy + \frac{1}{n} \delta_{ij} f(x).
\]  

**Proof.** As noted in the proof of Proposition 1, we may certainly write

\[
\partial_i \partial_j u(x) = \partial_i \int_{\mathbb{R}^n} f(y) \partial_j \Gamma_n(x-y) \, dy = \partial_i \int_{\mathbb{R}^n} f(x-y) \partial_j \Gamma_n(y) \, dy = \int_{\mathbb{R}^n} \partial_i f(x-y) \partial_j \Gamma_n(y) \, dy.
\]

The game of switching the convolution arguments between \( y \) and \( x-y \) is necessary because the derivatives \( \partial_i \) and \( \partial_j \) act on \( x \), not \( y \). Because \( \partial_j \Gamma_n \) is locally integrable, we have

\[
\partial_i \partial_j u(x) = \lim_{\varepsilon \to 0^+} \int_{|y| > \varepsilon} \partial_i f(x-y) \partial_j \Gamma_n(y) \, dy
\]

\[
= \lim_{\varepsilon \to 0^+} \partial_i \int_{|y| > \varepsilon} f(x-y) \partial_j \Gamma_n(y) \, dy
\]

\[
= \lim_{\varepsilon \to 0^+} \partial_i \int_{|x-y| > \varepsilon} f(y) \partial_j \Gamma_n(x-y) \, dy.
\]

We wish to once again move the derivative \( \partial_i \) inside the integral, but the region of integration now depends on \( x \). Accounting for this:

\[
\partial_i \int_{|x-y| > \varepsilon} f(y) \partial_j \Gamma_n(x-y) \, dy = \int_{|x-y| > \varepsilon} f(y) \partial_i \partial_j \Gamma_n(x-y) \, dy + \int_{|x-y| = \varepsilon} (\hat{x}_i \cdot \hat{n}) f(y) \partial_j \Gamma_n(x-y) \, dy.
\]

Hence

\[
\partial_i \partial_j u(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y| > \varepsilon} f(y) \partial_i \partial_j \Gamma_n(x-y) \, dy + \lim_{\varepsilon \to 0^+} \int_{|x-y| = \varepsilon} (\hat{x}_i \cdot \hat{n}) f(y) \partial_j \Gamma_n(x-y) \, dy.
\]

To complete the proof, we need to compute the second integral on the right hand side. As \( \varepsilon \to 0^+ \), we note that \( \hat{x}_i \cdot \hat{n} = O(1) \), \( f(y) = f(x) + O(\varepsilon) \), and \( \partial_j \Gamma_n(x-y) = O(\varepsilon^{-n+1}) \). The region of integration is a sphere of volume \( O(\varepsilon^{-n}) \). We therefore see that we may replace \( f(y) \) by \( f(x) \) in the integral to achieve the same limit. That is:

\[
\lim_{\varepsilon \to 0^+} \int_{|x-y| = \varepsilon} (\hat{x}_i \cdot \hat{n}) f(y) \partial_j \Gamma_n(x-y) \, dy = f(x) \lim_{\varepsilon \to 0^+} \int_{|x-y| = \varepsilon} (\hat{x}_i \cdot \hat{n}) \partial_j \Gamma_n(x-y) \, dy
\]

\[
= f(x) \lim_{\varepsilon \to 0^+} \int_{S^{n-1}(\varepsilon)} c_n z_i z_j |z|^{n+1} dA(z)
\]

\[
= \frac{1}{n} \delta_{ij} f(x).
\]