Let us first recall what we did last time. Last time, we considered \( u \in C^{2,\alpha}(B_1) \) and defined
\[
Nu = (\Delta u - u^3, u|_{\partial B_1}),
\]
where \( N : C^{2,\alpha}(\overline{B_1}) \to C^{\alpha}(\overline{B_1}) \oplus C^{2,\alpha}(\partial B_1) \). We’ll call these spaces \( X \) and \( Y \), so \( N : X \to Y \).

Then, \( N \) is a \( C^1 \) map,
\[
dN_u(v) = (\Delta v - 3u^2 v, v|_{\partial B})
\]
is an isomorphism \( X \to Y \) for all \( u \in X \). And as a corollary of the inverse function theorem, if \( F : X \to Y \) is \( C^1 \) and \( dF_x \) is an isomorphism, then the image of \( F \) contains a neighborhood of \( F(x) \).

**Proposition 1.** If \( u \in C^{2}(\overline{B}) \), and \( Nu = (f, \varphi) \), then
\[
(i) \|u\|_{C^0} \leq \|\varphi\|_{C^0} + \|f\|_{C^0} \\
(ii) \|u\|_{C^{2,\alpha}(B)} \leq g(\|f\|_{C^\alpha} + \|\varphi\|_{C^{2,\alpha}})
\]

An idea to prove (ii) is to first use global Schauder to get that
\[
\|u\|_{C^{2,\alpha}(\overline{B})} \lesssim \|
\Delta u\|_{C^\alpha(B)} + \|\varphi\|_{C^{2,\alpha}(\partial B)} \\
\leq \|u^3\|_{C^\alpha} + \|f\|_{C^\alpha} + \|\varphi\|_{C^{2,\alpha}}.
\]

Then, we might try to use that
\[
\|u^3\|_{C^\alpha} \leq \|u\|_{C^\alpha}^3 \leq (\epsilon \|u\|_{C^{2,\alpha}} + C_{\epsilon} \|u\|_{C^0})^3
\]
and rearrange. But we have an exponent of 3, so this doesn’t quite work. Instead, we take the inequality \( \|fg\|_{C^\alpha} \leq \|f\|_{C^\alpha} \|g\|_{C^\alpha} + \|f\|_{C^\alpha} \|g\|_{C^0} \) and we have
\[
\|u^3\|_{C^\alpha} \lesssim \|u\|_{C^\alpha}^2 \|u\|_{C^\alpha} \leq \|u\|_{C^0}^2(\epsilon \|u\|_{C^{2,\alpha}} + C_{\epsilon} \|u\|_{C^0})
\]
and now we can use rearrangement and the maximum principle to get the bounds that we want.

**Theorem 2.** \( N \) is surjective from \( X \to Y \).
Proof. Given \((f, \varphi) \in Y\), define

\[ \text{SOL} := \{ t \in [0, 1] : (tf, t\varphi) \in N(C^{2,\alpha}(B)) \}. \]

We want to show that \(1 \in \text{SOL}\). We already know that \(0 \in \text{SOL}\), so it will suffice to show that \(\text{SOL}\) is open and closed. \(\text{SOL}\) is open since if \(N_u = (t_0 f, t_0 \varphi)\), that \(dN_u\) is an isomorphism gives us that \(N(C^{2,\alpha}(B))\) contains a neighborhood of \((t_0 f, t_0 \varphi)\).

To show that \(\text{SOL}\) is closed, suppose that \(t_j \in \text{SOL}\) and \(t_j \to t_\infty\), and \(N u_j = (t_j f, t_j \varphi)\). By the proposition, \(\|u_j\|_{C^{2,\alpha}} \leq C\) uniformly in \(B\). By the Arzela-Ascoli theorem, \(u_j \to u_\infty\) in \(C^2\) for a subsequence. And \(N u_\infty = \lim N u_j = (f, \varphi)\). But

\[ \|u_\infty\|_{C^{2,\alpha}} \leq \limsup \|u_j\|_{C^{2,\alpha}} \leq C, \]

so the limit is in \(C^{2,\alpha}\). (We notice here that this does not say that \(u_j \to u_\infty\) in \(C^{2,\alpha}\), but says that \(u_j \to u_\infty\) in \(C^2\) and the limit is in \(C^{2,\alpha}\), which is good enough for our purposes. \(\square\)

Question: if \(\Delta u = 0\) on \(B\), \(u = \varphi\) on \(\partial B\), then is \(\|u\|_{C^1(B)} \lesssim \|\varphi\|_{C^1(\partial B)}\)?

Here’s a proof idea that doesn’t quite work. We know that \(\Delta \partial_i u = \partial_i \Delta u = 0\), so \(\partial_i u\) obeys the maximal principle. We want to say now that

\[ \|\partial_i u\|_{C^0} \leq \|\partial_i \varphi\|_{C^0} \leq \|\varphi\|_{C^1(\partial B)}, \]

but the first inequality does not hold since \(\varphi\) does not have derivatives in as many directions as \(u\) does (it is missing the directions normal to \(\partial B\)). This idea of bounding the derivatives in the normal direction will be important later on.

Next examples:

(i) \(\Delta u - |\nabla u|^2 = 0\) : this has good global regularity and we can solve the Dirichlet problem.

(ii) \(\Delta u - |\nabla u|^4 = 0\) : this has no global regularity and we can’t solve the Dirichlet problem.

Let us look at why the second case is bad. Take \(n = 1\). Then, we are looking for solutions to

\[ u'' - (u')^4 = 0. \]

If we take \(w = u'\), then we want to solve \(w' = w^4\). So \(w^{-4}w' = 1\). But \((w^{-3})' = -3w^{-4}w' = -3\). From this, we get that \(w(x)^{-3} = w(0)^{-3} - 3x\) and we have that

\[ w(x) = (w(0)^{-3} - 3x)^{-1/3}. \]

Now suppose that we want to solve \(u(0) = 0\) and \(u(1/3) = b\). For \(0 \leq b < H\), this is solvable but for \(b > H\), this is not solvable. We notice that if \(b \to H\), then then the norm of the boundary data
(the maximum of the values of the two points) is uniformly bounded, but \( |u'(1/3)| \to \infty \), and this is what causes our problem.

**Key Estimate:** If \( u \in C^2(\Omega) \), \( \Delta u - |\nabla u|^2 = 0 \), \( u = \varphi \) on \( \partial \Omega \), then

\[
\|\partial_n u\|_{C^0(\partial \Omega)} \leq C(\Omega) \|\varphi\|_{C^2(\partial \Omega)}.
\]

(Note: this also gives us that \( \|\partial u\|_{C^0(\partial \Omega)} \leq C(\Omega) \|\varphi\|_{C^2(\partial \Omega)} \).

**Proof Sketch:** We want to construct \( B : N \to \mathbb{R} \) such that

(i) \( B(x_0) = u(x_0) \)

(ii) \( B \geq u \) on \( \partial N \)

(iii) \( \Delta B - |\nabla B|^2 < 0 \)

(ii) and (iii) together will imply that \( B \geq u \) on \( N \). Then, \( \partial_n u(x_0) \leq \partial_n B(x_0) \).

**Proposition 3** (Comparison Principle). If

\[
Q u = \sum_{i,j} a_{ij}(\nabla u) \partial_i \partial_j u + b(\nabla u)
\]

is a quasilinear elliptic PDE, where \( a_{ij} \) are positive definite and \( a, b \in C^1 \) of \( \nabla u \), then if \( u, w \in C^2(\Omega) \), \( u \leq w \) on \( \partial \Omega \), \( Q u \geq Q w \) on \( \Omega \), then \( u \leq w \) on \( \Omega \)

**Proof of strict case.** We want to show that \( u - w \leq 0 \) on \( \Omega \) given that \( u - w \leq 0 \) on \( \partial \Omega \) and \( Q(u - w) > 0 \). Suppose \( x_0 \) is an interior maximum. Then, \( \nabla u(x_0) = \nabla w(x_0) = v_0 \). Then,

\[
\sum_{i,j} a_{ij}(v_0) \partial_i \partial_j (u - w)(x_0) > 0,
\]

but this is impossible at a local maximum. \( \square \)