Recall the following definition of $\Gamma(x)$:

$$\Gamma(x) = \begin{cases} \frac{1}{c_n |x|^{n-2}} & n \geq 3 \\ c_2 \log |x| & n = 2 \end{cases}$$

Note that derivatives of $\Gamma$ will trivially satisfy $|\nabla \Gamma| \approx |x|^{-n+1}$ and $|\partial^2 \Gamma| \approx |x|^{-n}$.

With this notation we have already proven:

**Prop 1:** If $u \in C^4_c(\mathbb{R}^n)$ and $\Delta u = f$ then $u = \Gamma * f$.

**Prop 2:** If $f \in C^2_c(\mathbb{R}^n)$ and $u = f * \Gamma$ then

$$\partial_i \partial_j u = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} f(y) \partial_i \partial_j \Gamma(x-y) dy + \frac{1}{n} \partial_i \partial_j f(x)$$

We now aim to prove

**THM 1:**

$$\sum_{u \in C^\infty} \frac{\|\partial_i \partial_j u\|_{L^\infty}}{\|\Delta u\|_{L^\infty}} = \infty$$

**THM 2** (Korn): For $0 < \alpha < 1$, $u \in C^2_c(\mathbb{R}^n)$,

$$[\partial_i \partial_j u]_{\alpha} \lesssim [\Delta u]_{\alpha}$$

Because of **Prop 1**, the general setup for proving **THM 1** is the following: Given $g$, we want to find an $f$ such that

1. $\|f\|_{L^\infty} = 1$
2. $[f * g](0) = \int f(y) g(-y) dy$ is very large

The best way to do this is clearly to take $f(y) = \text{sgn}(g(-y))$, so that $[f * g](0) = \int f(y) g(y) dy$.

For our situation, we have that $g = \partial_i \partial_j \Gamma$, and so $|g| \to \infty$. Smooth out our choice of $f$ so that it’s in $C^\infty_c(\mathbb{R}^n)$. Therefore, we’ve proven the following:

**Lemma:** For all $i, j, n, B > 0$, there exists $f_B \in C^\infty_c(B_1 \setminus \{0\})$ such that $u_B = f_B * \Gamma$, and $\|f_B\|_{L^\infty} \leq 1$ and $\partial_i \partial_j u_B(0) > B$.

**Proof of THM 1:** Let $w_B = u_B \eta_R$ where $\eta_R$ is a cutoff function which is 1 on $B_R$, 0 on $B_{2R}$, and satisfies $|\partial^k \eta_R| < C_k R^{-k}$. Then $\partial_i \partial_j w_B > B$, and

$$|\Delta w_B(x)| \leq |(\Delta u) \eta_B| + 2|\nabla u_B \cdot \nabla \eta_R| + |u_B \partial^2 \eta_R|$$

But since $|u_B| \lesssim |x|^{-n+2}$ and $|\nabla u_B| \lesssim |x|^{-n+1}$, and $|(\Delta u)| \lesssim 1$, and since derivatives of $\eta$ are bounded, we get that $|\Delta w_B(x)| \lesssim 1$.

Now let’s work towards a proof of Korn’s theorem. As setup, define

$$T_{\epsilon} f(x) = \int_{|x-y|>\epsilon} f(y) \partial_i \partial_j \Gamma(x-y) dy = f * K_{\epsilon}(x)$$
Then Korn’s Theorem can be equivalently expressed as

**THM 2’**: For \( f \in C^\alpha_c \), \( |T_\epsilon f|_\alpha \lesssim [f]_\alpha \).

Let’s take \( x_1, x_2 \in \mathbb{R}^n \), \( |x_1 - x_2| = d \) and normalize so \( [f]_\alpha = 1 \). Then we want to show that \( |T_\epsilon f(x_1) - T_\epsilon f(x_2)| \lesssim d^\alpha \).

There will be three typical examples to consider, and then a general case will break down into a sum of the examples.

**Ex 1**: Suppose that \( f \) is supported on \( B_{3d/4}(x_1) \cap B_{3d/4}(x_2) \). Then if \( y \in \text{supp} f \), \( d \geq |x_i - y| \geq d/4 \). Also, \( \text{supp } f \). Therefore, \( d \frac{1}{d} \int_{d/4 < |x_1 - y| < d} f(x) K(x_1 - y)dy \leq d^\alpha \int_{\text{Ann}} |x_1 - y|^{-n} dy \).

Now, \( |x_1 - y| \gtrsim d \) and \( \text{Vol}(\text{Ann}) \lesssim d^n \), so the whole integral is less than \( d^n \). This completes the inequality for this choice of \( f \).

**Ex 2**: Now suppose that \( f \) is supported on \( B_{d/2}(x_1) \). Around \( x_2 \) we can use the same argument as in **Ex 1** to get that \( |T_\epsilon f(x_2)| \lesssim d^\alpha \). However, around \( x_1 \) we need to use the cancellation of the kernel, namely, that

\[
0 = \int_{S_r} \partial_i \partial_j \Gamma(y)dy = \int_{S_r} K_r(y)dy
\]

Using this, we have that

\[
|T_\epsilon f(x_1)| \lesssim \int_{B_{d/2}(x_1)} \frac{|f(y) - f(x_1)|}{|x_1 - y|^\alpha} K_r(x_1 - y)|x_1 - y|^\alpha dy
\]

\[ \lesssim \int_{B_{d/2}(x_1)} |x_1 - y|^\alpha |K_r(x_1 - y)|dy \]

\[ \lesssim \int_{0 < |x_1 - y| < d/2} |x_1 - y|^{-n + \alpha} \lesssim d^\alpha \]

since that last integral is actually doable.
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