Chapter 4
Response of Materials to Surface Traction
Response of Materials

1. Deformation of the surface and subsurface

2. Fracture of Solids
Point force $P$ is applied at the origin of the cylindrical coordinate system ($r$, $\theta$, $z$).
Point Force Applied to a Semi-Infinite Elastic Solid (Boussinesq solution)

\[
\sigma_{rr} = \frac{P}{2\pi} \left[ (1 - 2\nu) \frac{1}{r^2} - \frac{z}{r^2 (r^2 + z^2)^{1/2}} - \frac{3r^2z}{(r^2 + z^2)^{3/2}} \right]
\]

\[
\sigma_{zz} = -\frac{3P}{2\pi} \frac{z^3}{(r^2 + z^2)^{5/2}}
\]

\[
\sigma_{\theta\theta} = \frac{(1 - 2\nu)P}{2\pi} \left[ -\frac{1}{r^2} + \frac{z}{r^2 (r^2 + z^2)^{1/2}} + \frac{z}{(r^2 + z^2)^{3/2}} \right]
\]

\[
\sigma_{rz} = -\frac{3P}{2\pi} \frac{rz^2}{(r^2 + z^2)^{5/2}}
\]
Point Force Applied to a Semi-Infinite Elastic Solid -- Resultant stress on face m

\[
\sigma^R = \sigma_{zz} + \sigma_{rz} = \frac{3P}{2\pi (r^2 + z^2)^2} \frac{z2}{2\pi (r^2 + z^2)^2} = \frac{3P \cos^2 \phi}{2\pi (r^2 + z^2)^2} = \frac{3P}{2\pi D^2}
\]

where

D is the diameter of the sphere passing through...
Point Force Applied to a Semi-Infinite Elastic Solid

\[ u = \frac{(1 - 2\nu)(1 + \nu)P}{2\pi Er} \left[ \frac{z}{(r^2 + z^2)^{1/2}} - 1 + \frac{r^2z}{(1 - 2\nu)(r^2 + z^2)^{3/2}} \right] \]

\[ v = 0 \]

\[ w = \frac{P}{2\pi E} \left[ \frac{(1 + \nu) z^2}{(r^2 + z^2)^{3/2}} + \frac{2(1 - \nu^2)}{(r^2 + z^2)^{1/2}} \right] \]

At \( z = 0 \), the radial and the vertical components of displacement become

\[ u = \frac{(1 - 2\nu)(1 + \nu)P}{2\pi Er} \]

\[ w = \frac{(1 - \nu^2)P}{\pi Er} \]

According to Eq. (4.4), the radial and vertical displacements at the surface decrease inversely with distance from the origin.
Trajectory of Principal stresses due to a Point Force

Diagram removed for copyright reasons.
Contour of Principal Normal Stresses

(\(\nu=0.25\))

Diagram removed for copyright reasons.
Angular Variation of Principal Stress Components in Boussinesq Field ($\nu=0.25$)

Diagram removed for copyright reasons.
Hertzian Contact

Figure 4.5 Two spherical bodies in contact. At zero load the contact occurs at a point $x = y = z_1 = z_2 = 0$. 

\[10\]
Hertzian Stress due to Spherical Indenters
(Normal Stress Distribution)

\[(\sigma_{zz})_{z=0} = -p = -p_0 \frac{(a^2 - r^2)^{1/2}}{a}\]
Hertzian Stress due to Spherical Indenters
(Normal Stress Distribution)

\[ a = \left[ \frac{3P}{16} \frac{k_1 + k_2}{\rho_1 + \rho_2} \right]^{1/3} = \left[ \frac{3P}{4} \frac{(k_1 + k_2) R_1 R_2}{R_1 + R_2} \right]^{1/3} \]

\[ p_0 = \frac{3P}{2\pi a^2} \]

where

\[ k_i = \frac{1 - \nu_i^2}{E_i} \quad i = 1 \text{ or } 2 \]

\[ \rho_i = \frac{1}{R_i} = \text{curvature of the sphere } i \]
Location of the Max. Radial Stress

Figure 4.6 Position of the maximum radial tensile stress
Stress Field due to Line Load

Figure 4.7 Semi-infinite elastic solid loaded by a concentrated load
Stress Field due to Line Load

\[
\sigma_{xx} = -\frac{2P}{\pi z} \cos^2 \theta \sin^2 \theta + \frac{2Q}{\pi z} \sin^3 \theta \cos \theta = \Lambda \sin^2 \theta
\]

\[
\sigma_{zz} = -\frac{2P}{\pi z} \cos^4 \theta + \frac{2Q}{\pi z} \cos^3 \theta \sin \theta = \Lambda \cos^2 \theta
\]

\[
\sigma_{xz} = -\frac{2P}{\pi z} \sin \theta \cos^3 \theta + \frac{2Q}{\pi z} \cos^2 \theta \sin^2 \theta = \Lambda \sin \theta \cos \theta
\]

where the angle \(\theta\) and the function \(\Lambda\) are given by

\[
\theta = \cos^{-1} \left( \frac{z}{(x^2 + z^2)^{1/2}} \right) \quad \Lambda = \frac{2}{\pi z} (-P \cos^2 \theta + Q \sin \theta \cos \theta)
\]
Semi-infinite elastic solid loaded by a elliptical distributed load

Figure 4.8 Semi-infinite elastic solid loaded by a elliptical distributed load. The maximum normal and tangential stresses are $p_0$ and $q_0$, respectively.
Semi-infinite elastic solid loaded by a elliptical distributed load

\[
\sigma_{zz} = \begin{cases} 
0 & \text{for } |x| > a \\
-p_0 \left(1 - \frac{x^2}{a^2}\right)^{1/2} & \text{for } |x| \leq a
\end{cases}
\]

\[
\sigma_{xz} = \begin{cases} 
0 & \text{for } |x| > a \\
q_0 \left(1 - \frac{x^2}{a^2}\right)^{1/2} & \text{for } |x| \leq a
\end{cases}
\]

The maximum normal and tangential stresses are \(p_0\) and \(q_0\), respectively.
Semi-infinite elastic solid loaded by a elliptical distributed load

\[
\sigma_{xx} = \frac{2q_0}{\pi az} \int_{-a}^{a} (a^2 - \xi^2)^{1/2} \frac{(x - \xi)z}{[z^2 + (x - \xi)^2]^2} \, d\xi \\
- \frac{2P_0}{\pi az} \int_{-a}^{a} (a^2 - \xi^2)^{1/2} \frac{z^2(x - \xi)^2}{[z^2 + (x - \xi)^2]^2} \, d\xi
\]

\[
\sigma_{zz} = \frac{2q_0}{\pi az} \int_{-a}^{a} (a^2 - \xi^2)^{1/2} \frac{(x - \xi)z^3}{[z^2 + (x - \xi)^2]^2} \, d\xi \\
- \frac{2P_0}{\pi az} \int_{-a}^{a} (a^2 - \xi^2)^{1/2} \frac{z^4}{[z^2 + (x - \xi)^2]^2} \, d\xi
\]

\[
\sigma_{xz} = \frac{2q_0}{\pi az} \int_{-a}^{a} (a^2 - \xi^2)^{1/2} \frac{(x - \xi)^2z^2}{[z^2 + (x - \xi)^2]^2} \, d\xi \\
- \frac{2P_0}{\pi az} \int_{-a}^{a} (a^2 - \xi^2)^{1/2} \frac{z^3(x - \xi)}{[z^2 + (x - \xi)^2]^2} \, d\xi
\]

The maximum normal and tangential stresses are \( p_0 \) and \( q_0 \), respectively.
Semi-infinite elastic solid loaded by a elliptical distributed load

\[ \sigma_{xx} = \frac{q_0}{\pi} \left[ (2x^2 - 2a^2 - 3z^2)\psi + 2\pi \frac{x}{a} + 2(a^2 - x^2 - z^2)\frac{x}{a}\psi \right] - \frac{P_0}{\pi} z \left( \frac{a^2 + 2x^2 + 2z^2}{a} - \frac{2}{a} - 3x\psi \right) \]

\[ \sigma_{zz} = \frac{q_0}{\pi} z^2\psi - \frac{P_0}{\pi} z(a\overline{\psi} - x\psi) \]

\[ \sigma_{xz} = \frac{q_0}{\pi} \left[ \left( a^2 + 2x^2 + 2z^2 \right)\frac{z}{a}\overline{\psi} - 2\pi \frac{z}{a} - 3xz\psi \right] - \frac{P_0}{\pi} z^2\psi \]

in which

\[ \psi = \frac{\pi}{k_1 (k_2/k_1)^{1/2}} \frac{1 - (k_2/k_1)^{1/2}}{2(k_2/k_1)^{1/2} + [(k_1 + k_2 - 4a^2)/k_1]^{1/2}} \]  

\[ \overline{\psi} = \frac{\pi}{k_1 (k_2/k_1)^{1/2}} \frac{1 + \left( \frac{k_2}{k_1} \right)^{1/2}}{2(k_2/k_1)^{1/2} + [(k_1 + k_2 - 4a^2)/k_1]^{1/2}} \]

\[ k_1 = (a + x)^2 + z^2 \]

\[ k_2 = (a - x)^2 + z^2 \]

The maximum normal and tangential stresses are \( p_0 \) and \( q_0 \), respectively.
Semi-infinite elastic solid loaded by a elliptical distributed load

\[
\begin{align*}
\sigma_{xx} &= \frac{q_0}{\pi} \left[ (2x^2 - 2a^2 - 3z^2)\psi + 2\pi \frac{x}{a} + 2(a^2 - x^2 - z^2)\frac{x}{a}\psi \right] \\
&\quad - \frac{P_0}{\pi} z \left( \frac{a^2 + 2x^2 + 2z^2}{a} \psi - \frac{2}{a} - 3x\psi \right) \\
\sigma_{zz} &= \frac{q_0}{\pi} z^2\psi - \frac{P_0}{\pi} z(a\overline{\psi} - x\psi) \\
\sigma_{xz} &= \frac{q_0}{\pi} \left[ (a^2 + 2x^2 + 2z^2)\frac{z}{a}\overline{\psi} - 2\pi \frac{z}{a} - 3xz\psi \right] - \frac{P_0}{\pi} z^2\psi
\end{align*}
\]

(4.14)

in which

\[
\psi \equiv \frac{\pi}{k_1} \frac{1 - (k_2/k_1)^{1/2}}{(k_2/k_1)^{1/2}[2(k_2/k_1)^{1/2} + [(k_1 + k_2 - 4a^2)/k_1]]^{1/2}}
\]

(4.15)

\[
\overline{\psi} \equiv \frac{\pi}{k_1} \frac{1 + \left(\frac{k_2}{k_1}\right)^{1/2}}{(k_2/k_1)^{1/2}[2(k_2/k_1)^{1/2} + [(k_1 + k_2 - 4a^2)/k_1]]^{1/2}}
\]

\[
k_1 \equiv (a + x)^2 + z^2 \\
k_2 \equiv (a - x)^2 + z^2
\]

The maximum normal and tangential stresses are \( p_0 \) and \( q_0 \), respectively.
Semi-infinite elastic solid loaded by a elliptical distributed load

The maximum normal and tangential stresses are $p_0$ and $q_0$, respectively.

\[
\sigma_{xx} = \begin{cases} 
2q_0 \left[ \frac{x}{a} - \left( \frac{x^2}{a^2} - 1 \right)^{1/2} \right] & \text{for } x \geq a \\
2q_0 \left[ \frac{x}{a} + \left( \frac{x^2}{a^2} - 1 \right)^{1/2} \right] & \text{for } x \leq -a \\
2q_0 \frac{x}{a} - p_0 \left( 1 - \frac{x^2}{a^2} \right)^{1/2} & \text{for } |x| \leq a 
\end{cases}
\]
Contour and Variation of Stress

Graphs removed for copyright reasons.
See Figures 4.9 through 4.15 in [Suh 1986].
Crack Growth of Hertzian Crack

(soda-lime glass in air)

(a)

(b)

(c)

(d)

Figure 4.16
Surface Traces of Hertzian Cracks on Surface of Silicon

Photos removed for copyright reasons.
See Figure 4.18 in [Suh 1986].
Crack Patterns on Sod-lime Glass Produced by Sliding Tungsten Carbide Sphere

Photos removed for copyright reasons. See Figure 4.17 in [Suh 1986].
“Star” Crack in Soda-Lime Glass produced by Conical Indenter

Photos removed for copyright reasons. See Figure 4.19 in [Suh 1986].
Sequence of Crack Formation and Growth Events during Loading and Unloading

Photos removed for copyright reasons.
See Figure 4.20 in [Suh 1986].