Example 2: Particle on String Pulled Through Hole

Figure 1: Particle on string pulled through hole. Tabletop with hole B. A string comes out with an attached mass. The particle is traveling around with an angular velocity $\dot{\theta}$. Figure by MIT OCW.


Pull string through hole at B such that:

\[
\begin{align*}
  r(t_0) &= L & \frac{dr}{dt}(t_0) &= 0 \\
  r(t_1) &= L/2 & \frac{dr}{dt}(t_1) &= 0
\end{align*}
\]

If $\dot{\theta}(t_0) = \dot{\theta}_0$, what is $\dot{\theta}(t_1) = \dot{\theta}_1$?
Discussion

If we use linear momentum, will need to describe forces between \( m \) and string. Thinking about angular momentum about the point \( B \):

\[
\tau_B = \dot{h}_B + \omega_B \times m \mathbf{v} \quad \text{Angular momentum principle}
\]

\[
h_B = \mathbf{r} \times m \mathbf{v} \quad \text{Angular Momentum}
\]

Now:

\[
\tau_B = \mathbf{r} \times \mathbf{F} \quad \text{Forces acting on particle \( \tau_B = 0 \) because \( \mathbf{r} \parallel \mathbf{F} \)}
\]

\[
\tau_B = \dot{h}_B + \omega_B \times m \mathbf{v} \Rightarrow \text{Angular momentum about } B \text{ is constant } \dot{h}_B = 0.
\]

\[
\tau_B = 0 \text{ (from above)}
\]

\[
\omega_B = 0 \text{ because } B \text{ is not moving}
\]

\[
\therefore \dot{h}_B = \text{Constant}
\]

In Cartesian Coordinates

\[
\mathbf{r} = r \cos \theta \hat{i} + r \sin \theta \hat{j}
\]

\[
\mathbf{p} = m \mathbf{v} = m \dot{\mathbf{r}} = -mr \dot{\theta} \sin \theta \hat{i} + mr \dot{\theta} \cos \theta \hat{j}
\]

a. \( \dot{h}_B(t_0) = \mathbf{r} \times \mathbf{p} = LmL \dot{\theta}_0 \hat{k} \) (\( \hat{k} \) is unit vector in z-direction: out of page).

b. \( \dot{h}_B(t_1) = \frac{L^2}{2} m \dot{\theta}_1 \hat{k} \)

Setting (a) = (b): \( \dot{\theta}_1 = 4\dot{\theta}_0 \), and velocity of particle \( v_1 = 2v_0 = \frac{L}{2} 4\dot{\theta}_0 = 2L\dot{\theta}_0 \).

Energy is not conserved: why? The pulling force (tension) does work.

Dynamics of systems of particles

Forces on each particle may be composed as follows

\[
\mathbf{F} = \mathbf{F}^{\text{ext}} + \mathbf{F}^{\text{int}}
\]
Dynamics of systems of particles

Figure 2: Dynamics of systems of particles. Figure by MIT OCW.

\( F_i \): Resultant force acting on \( m_i \)

\( F_{\text{ext}} \): External forces (e.g. gravity)

\( F_{\text{int}} \): Internal forces between particles (e.g. charge attraction)

\[
\sum_{j=1}^{n} F_{ij} \text{ Force on particle } i \text{ due to particle } j
\]

Newton’s Third Law

\[
F_{ij} = -F_{ji}
\]

Thus:

\[
\sum_{i=1}^{n} F_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} = 0
\]

Sum of all internal forces is zero, therefore:

\[
\sum_{i=1}^{n} F_{int} = 0
\]

Total internal torques is also zero: demonstrate by considering an arbitrary pair of particles.
Dynamics of systems of particles

Figure 3: Arbitrary pair of particles subject to individual forces. Figure by MIT OCW.

\[ \tau_B = \mathbf{r}_{i/B} \times \mathbf{f}_{ij} + \mathbf{r}_{j/B} \times \mathbf{f}_{ji} = (\mathbf{r}_{i/B} - \mathbf{r}_{j/B}) \times \mathbf{f}_{ij} \]

but \((\mathbf{r}_{i/B} - \mathbf{r}_{j/B} \parallel \mathbf{f}_{ij})\)

∴ \(\tau_{int}^B = 0\) No net internal torque

Center of mass

\[ \mathbf{r}_c = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{M} \]

\(M\): Total Mass of System

Note that this relation can also be written as \(\sum_{i=1}^{n} m_i (\mathbf{r}_i - \mathbf{r}_c) = 0\) i.e. center of mass is the point about which the total mass moment is zero.

Newton’s Laws for Systems of Particles

(Williams: C-1 to C-3.6)

Derivation needed to prevent mistakes in applying the laws later. Will be able to use results for rigid bodies.
Linear Momentum Principle (for a single particle)

\[ F_i = \frac{d}{dt}p_i \]

\( F_i \): Total Force on particle \( i \)
\( p_i \): Linear momentum of particle \( i \)

\[ \sum_{i=1}^{n} F_{ext}^i + \sum_{i=1}^{n} F_{int}^i = \frac{d}{dt} \sum_{i=1}^{n} p_i = \frac{d}{dt} \vec{p} \]

\( F_{int}^i = 0 \)

\( \vec{F}_{ext} = \frac{d}{dt} \vec{p} \)

\( \vec{F}_{ext} \): Sum of \( F \) external for whole system.

Note that total linear momentum:

\[ \vec{p} = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m_i v_i = M\vec{v}, \text{ where } \vec{v} = \dot{\vec{r}} = \frac{d}{dt} \sum_{i=1}^{n} m_i r_i \]

If \( \sum_{i=1}^{n} F_{ext}^i = \vec{F}_{ext} = 0 \Rightarrow \vec{p} = \text{constant}; \text{ therefore, } \vec{v} = \text{constant}. \)

Example: You have a ball as a ice skater. Throw object, both ball and skater move, but center of mass stays the same, does not move.

Angular Momentum Principle

From Newton II \( F_i = \frac{d}{dt}p_i \)

Torque:

\[ \vec{r}_i \times \vec{F}_i = \vec{r}_i \times \frac{d}{dt}p_i \]

Sum over all particles.

\[ \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_{ext}^i = \sum_{i=1}^{n} \vec{r}_i \times \frac{d}{dt}p_i \]

Later will need vectors to center of mass.

\[ \sum_{i=1}^{n} \vec{r}_{ext}^i_B = \text{Sum of all external torques about } B \]
Dynamics of systems of particles

\[ \sum_{i=1}^{n} \tau_{ext}^i = \sum_{i=1}^{n} r_i' \times \frac{d}{dt} p_i = \sum_{i=1}^{n} \frac{d}{dt} (r_i' \times p_i) - \sum_{i=1}^{n} \frac{d}{dt}(r_i' p_i) \]

\[ \sum_{i=1}^{n} \omega_B = \frac{d}{dt} \sum_{i=1}^{n} \mu_i B - \sum_{i=1}^{n} \frac{d}{dt}(r_i - r_B) \times p_i \]

\[ \omega_B = \frac{d}{dt} H_B - \sum_{i=1}^{n} \mu_i \times p_i + \sum_{i=1}^{n} \mu_i B \times p_i \]

\( \mu_B \) is the same for each \( p_i \).

\[ \sum_{i=1}^{n} \mu_B \times p_i = \mu_B \times \sum_{i=1}^{n} p_i = \mu_B \times p \]

So, finally we have:

\[ \tau_{ext}^i = \frac{d}{dt} H_B + \mu_B \times p \]

\( \tau_{ext}^i \): Total External Torque

\( \frac{d}{dt} H_B \): Total Angular Momentum

\( \mu_B \times p \): Total Linear Momentum

Next time: Consequences of this expression and work-energy principle.

Figure 4: A system of particles subject to a force. Figure by MIT OCW.