Kinetics of Rigid Bodies

Angular Momentum Principle for a Rigid Body

\[ H_B = \sum_i r_i' \times m_i (\omega_i + \omega \times \rho_i) \]

After some steps (see Lecture 8):

\[ H_B = r_c' \times P + \sum_i m_i \rho_i \times \omega \times \rho_i \]

We now use:

\[ a \times b \times c = (a \cdot c) b - (a \cdot b) c \]

\[ \rho_i \times \omega \times \rho_i = \rho_i^2 \omega - (\omega \cdot \rho_i) \rho_i \]

\[ = \rho_i^2 \omega \]
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For 2-D motion, \( \omega \cdot \rho = 0 \) because the vectors are \( \perp \). For 3-D, this term does not have to be 0.

\[
H_B = r_c' \times P + \sum_i m_i \rho_i^2 \omega_i
\]

\[
= r_c' \times P + I_c \omega
\]

\( I_c \): Moment of Inertia. \( I_c = \sum_i m_i \rho_i^2 \) (Intrinsic Property of Rigid Body)

Example:

Figure 2: Hoop and Disc, both with mass \( M \). Figure by MIT OCW.

\[
H_B = r_c' \times P + I_c \omega
\]

If one takes angular momentum about the center of mass:

\[
H_c = I_c \omega
\]

(Angular Momentum about B) = (Angular Momentum about C) + (Moment of Linear Momentum about B)

Therefore:

\[
H_B = H_c + r_c' \times P
\]

**Special Case of Fixed Axis of Rotation about B**

i.e. \( \omega_c = \omega_B + \omega \times r_c' \)
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Figure 3: Rigid body which pivots about B. Figure by MIT OCW.

\[ H_B = H_C + r_C \times m(\omega \times r_C) \]
\[ = H_C + mr_C^2 \omega \]
\[ = (I_C + mr_C^2)\omega = I_B \omega \]

\[ I_B = I_C + mr_C^2 \quad \text{Parallel Axis Theorem} \]

Only do this if the \( v_B = 0 \) and \( v_C = (\omega \times r_C) \)

Finally:

\[ \tau_{ext}^B = \frac{d}{dt} H_B + v_B \times P \quad (1) \]
\[ H_B = H_C + r_C \times P \quad (2) \]
\[ H_C = I_C \omega \quad (3) \]
\[ I_C = \sum m_i r_i^2 \quad (4) \]

Equations (1) to (4) are always true.

Special Cases

1. \( B = C \Rightarrow r_C = 0; \quad v_B \parallel P \)

Start by thinking about motion around center of mass.

\[ \tau_{ext}^B = \frac{d}{dt} H_C \quad \text{and} \quad H_C = I_C \omega \]
Cue hitting a pool ball

2. B is a stationary point and fixed in the body.

\[ \tau_{ext}^{B} = \frac{d}{dt}H_{B} \quad \text{and} \quad H_{B} = I_{B}\omega \quad \text{where} \quad I_{B} = I_{C} + mr_{C}^{2} \]

What do we need to do still?
Calculating moments of inertia ⇒ Recitation 5
Work-Energy Principle

Cue hitting a pool ball

A pool ball of radius R and mass M is at rest on a horizontal table. It is set in motion by a sharp horizontal impulse J provided by the cue. Determine the height above the ball’s center that the cue should strike so that the subsequent motion is rolling without slipping.

\[ y_{C} = \text{constant} = R \]
\[ v_{C} = \omega R \quad \text{(or} \quad x_{c} = R\theta) \]

1 Degree of Freedom. (Use \( x_{C} \) or \( \theta \)).
Kinetics: Free Body Diagrams

![Free Body Diagram](image)

Figure 5: Free Body Diagram of Cue Ball. Figure by MIT OCW.

Impulse force that provides impulse $\mathbf{I}$

$$\mathbf{I} = \int_{0^-}^{0^+} F \, dt = \int_{0^-}^{0^+} J \delta(t) \, dt \text{ i.e. } \mathbf{F} = \mathbf{J} \delta(t)$$

(i) Linear Momentum Principle

$$\mathbf{F}^{\text{ext}} = \frac{d}{dt} \mathbf{P}$$

y-direction: $C$ always at same height. $N = mg$ so no vertical motion of $C$.

x-direction: $F = J\delta(t) = \frac{d}{dt} Mv_C$.

Integrate both sides

$$\int_{0^-}^{0^+} F \, dt \int_{0^-}^{0^+} J \delta(t) \, dt = \int_{0^-}^{0^+} \frac{d}{dt} M v_C \, dt$$

$$J = M v_C(0^+) - M v_C(0^-)$$

$J$: Momentum Imparted

$$J = M v_C(0^+)$$  \hspace{1cm} (5)

Angular Momentum Principle About $C$

Taking momentum about $C$ simplifies equations
Figure 6: Angular Momentum Principle about C applied to Cue Ball. Figure by MIT OCW.

\[
\tau_{ext} = \frac{d}{dt}H_c \quad \text{and} \quad H_C = I_C \omega
\]

\[
r_F \times F = \frac{d}{dt}I_C \omega
\]

\[
-F h \dot{e}_z = -I_C \frac{d}{dt} \dot{e}_z
\]

\[
\int_{0^-}^{0^+} F h dt = \int_{0^-}^{0^+} I_C \frac{d}{dt} \omega dt
\]

\[
\int_{0^-}^{0^+} J \delta(t) dt = I_C \omega(0^+) - I_C \omega(0^-)
\]

\[I_C \omega(0^-) = 0 \quad \text{because} \quad \omega(0^-) = 0.\]

\[J h = I_C \omega(0^+)\]

Impulsive torque about center of mass = Change of angular momentum caused by the torque
Satisfying Constraints

If there is no slip, one needs \( \omega(0^+)R = v_C(0^+) \)

![Diagram of Cue Ball moving](image)

Figure 7: Diagram of Cue Ball moving. This diagram demonstrates how to satisfy geometric constraints of movement. Figure by MIT OCW.

\[
J = \frac{I_C}{h} \frac{v_C(0^+)}{R}
\]

Can eliminate \( J \) from Equation 5 and Equation 6

\[
M v_C(0^+) = \frac{I_C}{h} \frac{v_C(0^+)}{R} \Rightarrow h = \frac{I_C}{mR} 
\]

For a sphere:

\[
I_C = \frac{2}{5} mR^2 \Rightarrow h = \frac{2}{5} R
\]

\( h \): Independent of mass of sphere. Independent of force applied.