Example: Cart with Pendulum and Spring (Continued)

Recall:

![Cart with Pendulum and Spring Diagram]

Figure 1: Cart with pendulum and spring. Figure by MIT OCW.

Equations of Motion

\[(M + m)\ddot{x} + m(\dot{s}\cos{\theta} + s\dot{\theta}\cos{\theta} - s\dot{\theta}^2\sin{\theta}) = 0 \quad (1)\]

\[s\ddot{\theta} + 2\dot{s}\dot{\theta} + \ddot{x}\cos{\theta} + g\sin{\theta} = 0 \quad (2)\]

\[m\ddot{s} + m\ddot{x}\sin{\theta} - ms\dot{\theta}^2 - mg\cos{\theta} + k(s - l) = 0 \quad (3)\]
Example: Cart with Pendulum and Spring (Continued)

**Equilibria**

\[ \ddot{s} = \dot{s} = \ddot{\theta} = \dot{\theta} = \ddot{x} = \dot{x} = 0 \]

Set all variables to 0 except for position variables.

**Configuration:**

\[ \theta_0 = 0, \ s_0 = l + \frac{mg}{k} \] (Stable)
\[ \theta_0 = \pi, \ s_0 = l - \frac{mg}{k} \] (Stable or unstable? Expect it to be unstable based on physical intuition.)

\[ x \] for both can be any value.

**Linearize Equations (1), (2), and (3) about \( \theta_0 = \pi \) and \( s_0 = l - mg/k \)**

\[ \theta = \theta_0 + \phi, \ s = s_0 + \epsilon \]
\[ \dot{\theta} = \dot{\phi}, \ \ddot{\theta} = \ddot{\phi}, \ \dot{s} = \dot{\epsilon}, \ \ddot{s} = \dddot{\epsilon} \]

**Taylor Series**

\[
\cos(\theta_0 + \phi) = \cos \theta_0 + \frac{d}{d\theta} \cos \theta \bigg|_{\theta_0} \phi + \ldots \approx -1 + 0 \text{ for } \theta = \pi
\]

\[
\sin(\theta_0 + \phi) = \sin \theta_0 + \frac{d}{d\theta} \sin \theta \bigg|_{\theta_0} \phi = 0 - \phi \text{ for } \theta = \pi
\]

**Linearization**

\[
(M + m)\ddot{x} + m[\dot{\epsilon}(-\phi) + 2\dot{\epsilon}\dot{\phi}(-1) + (s_0 + \epsilon)\ddot{\phi}(-1) - (s_0 + \epsilon)\dot{\phi}^2(-\phi)] = 0
\]

\[
(M + m)\ddot{x} - ms_0\ddot{\phi} = 0 \tag{1L_\phi}
\]

\[
(s_0 + \epsilon)\ddot{\phi} + 2\dot{\epsilon}\ddot{\phi} + \dot{x}(-1) + g(-\phi) = 0
\]

\[
s_0\ddot{\phi} - \dddot{x} - g\phi = 0 \tag{2L_\phi}
\]

\[
m\dddot{x} + m\ddot{x}(-\phi) - m(s_0 + \epsilon)\dot{\phi}^2 - mg(-1) + k(s_0 + \epsilon + l) = 0
\]
Example: Cart with Pendulum and Spring (Continued)

\[ s_0 = l - \frac{mg}{k} \]

\[ m\ddot{\epsilon} + k\epsilon = 0 \quad (3L_\phi) \]

Solution and Analysis

From (1L_\phi):

\[ \ddot{x} = \frac{m s_0}{(M + m)} \dot{\phi} \]

Substitute in (2L_\phi).

\[ s_0 \ddot{\phi} - \frac{m s_0}{(M + m)} \dot{\phi} - g \phi = 0 \]

\[ \ddot{\phi} - \frac{g(M + m)}{M s_0} \phi = 0 \]

Solutions are of form \( \phi = \phi_0 e^{\lambda t} \). \( \ddot{\phi} = \lambda^2 \phi \).

\[ \lambda^2 - g \frac{(M + m)}{M s_0} = 0 \]

Finally:

\[ \lambda = \pm \sqrt{\frac{g(M + m)}{M s_0}} \]

Predicts exponential growth of \( \theta \) disturbances in time, therefore unstable.

Equation (3L_\phi) still predicts that small stretches of the spring lead to oscillations.
Spinning Hoop with Sliding Mass

Figure 2: Spinning hoop with sliding mass. The hoop of radius, $a$ rotates. The mass, $m$ slides around the hoop. Figure by MIT OCW.

Massless ring - Frictionless
Rotating about the vertical axis
Center O with Radius $a$
$m$ slides on hoop - 2 degrees of freedom. If $m$ were a free particle, 3 degrees of freedom.
Torque, $\tau$ about $z$ axis.

Generalized Coordinates and Generalized Forces
Two generalized coordinates: $\theta$, $\phi$

$$\Xi_{\theta} = 0$$

What is the work done with a small change $\theta$? None. Only gravity.

$$\Xi_{\phi} = \tau$$

External torque applied.
Kinematics

![Figure 3: Kinematic diagram of sliding mass on hoop. Figure by MIT OCW.](image)

⊥ components sliding on hoop, rotating into page.

**Lagrangian**

\[
L = T - V
\]

\[
T = \frac{1}{2} m v_{\text{particle}}^2 = \frac{1}{2} m (a^2 \dot{\theta}^2 + a^2 \sin^2 \theta \dot{\phi}^2)
\]

\[
V = - mga \cos \theta
\]

\[
L = T - V = \frac{1}{2} m (a^2 \dot{\theta}^2 + a^2 \sin^2 \theta \dot{\phi}^2) - mga \cos \theta
\]

**Equations of Motion**

θ:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \Xi_\theta
\]

\[
\frac{\partial L}{\partial \dot{\theta}} = ma^2 \dot{\theta}
\]

\[
\frac{\partial L}{\partial \theta} = ma^2 \sin \theta \cos \theta \dot{\phi}^2 - mga \sin \theta
\]

\[
ma^2 \ddot{\theta} - ma^2 \sin \theta \cos \theta \dot{\phi}^2 + mga \sin \theta = 0
\]  

(4)

φ:

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = \Xi_\phi
\]
\[
\frac{\partial L}{\partial \phi} = ma^2 \sin^2 \theta \dot{\phi} \\
\frac{\partial L}{\partial \dot{\phi}} = 0 \\
\Xi = \tau
\]

\[
\frac{d}{dt}(ma^2 \sin^2 \theta \dot{\phi}) = \tau
\]

**Modification: Add Controller**

Imagine a controller that keeps \( \dot{\phi} \) a constant.

Assume \( \dot{\phi} = \text{constant} = \Omega \).

\[
\tau = \frac{d}{dt}(ma^2 \sin \theta \Omega)
\]

Controller measures \( \theta \) and \( \dot{\theta} \), then sets \( \tau \) so that \( \Omega \) is constant.

Assume that this equation is always satisfied by controller.

Equation (4) becomes:

\[
\ddot{\theta} - \sin \theta \cos \theta \Omega^2 + \frac{g}{a} \sin \theta = 0
\]

**Equilibrium Points**

Equilibria - must use original nonlinear equations to determine equilibrium points.

\[
\dot{\theta} = \ddot{\theta} = 0 \\
- \sin \theta \cos \theta \Omega^2 + \frac{g}{a} \sin \theta = 0
\]

Physical intuition tells us some equilibria should fall at \( \theta = 0 \) or \( \theta = \pi \).

\[
\sin \theta \left( \frac{g}{a} - \cos \theta \Omega^2 \right) = 0 \quad (5)
\]

\[
\theta_0 = 0, \pi \quad (\sin \theta = 0)
\]
Spinning Hoop with Sliding Mass

Three equilibrium positions from the equation.

\[ \cos \theta = \frac{g}{a \Omega^2}, \quad \theta_e = \arccos \frac{g}{a \Omega^2} \]

Balance between gravity and centripetal force (normal force from hoop)

Note that the solution \( \theta_e = \arccos \frac{g}{a \Omega^2} \) only exists, provided \( \frac{g}{a \Omega^2} < 1 \) or \( \theta_e \) equilibrium only exists when rotation is fast enough (\( \Omega^2 \geq \frac{g}{a} \)).

Stability

**Stability around** \( \theta_e = \arccos(g/a \Omega^2) \)

Look at \( \theta_e = \arccos \frac{g}{a \Omega^2} \)

\[ \theta = \theta_e + \epsilon, \quad \dot{\theta} = \dot{\epsilon}, \quad \ddot{\theta} = \ddot{\epsilon} \]

\[ \ddot{\epsilon} - \sin(\theta_e + \epsilon) \cos(\theta_e + \epsilon) \Omega^2 + \frac{g}{a} \sin(\theta_e + \epsilon) = 0 \]

Use angle addition formulas to expand (an alternative to Taylor series expansion):

\[ \ddot{\epsilon} - (\sin \theta_e \cos \epsilon + \cos \theta_e \sin \epsilon)(\cos \theta_e \cos \epsilon - \sin \theta_e \sin \epsilon)\Omega^2 + \frac{g}{a}(\sin \theta_e \cos \epsilon + \cos \theta_e \sin \epsilon) = 0 \]

\[ \ddot{\epsilon} - (\sin \theta_e \cos \theta_e \cos^2 \epsilon - \sin^2 \theta_e \cos \epsilon \sin \epsilon + \cos^2 \theta_e \sin \epsilon \cos \epsilon - \sin^2 \epsilon \sin \theta_e \cos \theta_e)\Omega^2 \]

\[ + \frac{g}{a}(\sin \theta_e \cos \theta_e + \cos \theta_e \epsilon) = 0 \]

(6)

We approximate:

\[ \cos \epsilon \approx 1 \]

\[ \sin \epsilon \approx \epsilon \]

\[ \ddot{\epsilon} - \Omega^2 \sin \theta_e \cos \theta_e + (\sin^2 \theta_e - \cos^2 \theta_e)\Omega^2 \cdot \epsilon + \frac{g}{a} \sin \theta_e + \frac{g}{a} \epsilon \cos \theta_e = 0 \]

Terms with no \( \epsilon \) (no perturbation variable) are a restatement of the equilibrium configuration you already found.

\[ \frac{g}{a} \sin \theta_e - \Omega^2 \sin \theta_e \cos \theta_e \rightarrow \sin \theta_e \left( \frac{g}{a} - \Omega^2 \cos \theta_e \right) \]

So those terms cancel out by the equilibrium condition shown in 6.

\[ \ddot{\epsilon} - \Omega^2 (\cos^2 \theta_e - \sin^2 \theta_e) \epsilon + \frac{g}{a} \epsilon \cos \theta_e = 0 \]

But \( \cos \theta_e = \frac{g}{a \Omega^2} \), so
\[ \ddot{\epsilon} + \Omega^2 \sin^2 \theta \epsilon = 0 \]

Stable, because this is a positive sign. \( \sin \) or \( \cos \) solutions. Oscillations. Stable.

Next time: Equilibrium points \( \theta = 0, \pi \).