REVIEW Lecture 15:

• Finite Volume Methods
  – Integral and conservative forms of the cons. laws
  – Introduction
  – Approximations needed and basic elements of a FV scheme
    • Grid generation ⇒ Time-Marching
    • FV grids: Cell centered (Nodes or CV-faces) vs. Cell vertex; Structured vs. Unstructured
    • Approximation of surface integrals (leading to symbolic formulas)
    • Approximation of volume integrals (leading to symbolic formulas)
    • Summary: Steps to step-up a FV scheme
  – One Dimensional examples
    • Generic equation:
      \[
      \frac{d}{dt} (\Delta x \bar{\Phi}_j) + f_{j+1/2} - f_{j-1/2} = \int_{x_{j-1/2}}^{x_{j+1/2}} s_\phi(x,t) \, dx
      \]
    • Linear Convection (Sommerfeld eqn): convective fluxes
      – 2\textsuperscript{nd} order in space
TODAY (Lecture 16):
FINITE VOLUME METHODS

• Summary: Steps to step-up a FV scheme
• Examples: One Dimensional examples
  – Generic equations
  – Linear Convection (Sommerfeld eqn): convective fluxes
    • 2nd order in space, 4th order in space, links to CDS
  – Unsteady Diffusion equation: diffusive fluxes
    • Two approaches for 2nd order in space, links to CDS
• Approximation of surface integrals and volume integrals revisited
• Interpolations and differentiations
  – Upwind interpolation (UDS)
  – Linear Interpolation (CDS)
  – Quadratic Upwind interpolation (QUICK)
  – Higher order (interpolation) schemes
References and Reading Assignments


One-Dimensional Example I
Linear Convection (Sommerfeld) Eqn, Cont’d

• The resultant linear algebraic system is circulant tri-diagonal (for periodic BCs)

\[
\frac{d \Phi}{dt} + \frac{c}{2\Delta x} B_p(-1,0,1)\Phi = 0
\]

• This is as the 2\textsuperscript{nd} order CDS!, except that it is written in terms of cell averaged values instead of values at FD nodes/points

  – It is also 2\textsuperscript{nd} order in space

  – Has same properties as classic CDS for

\[
\frac{\partial \phi(x,t)}{\partial t} + \frac{\partial c \phi(x,t)}{\partial x} = 0
\]

• Non-dissipative (check Fourier analysis or eigenvalues of $B_p$ which are imaginary), but can provide oscillatory errors

• Stability (recall tables for FD schemes, linear convection eqn.) of time-marching

  – If centered in time, centered in space, explicit: stable with CFL condition: 

\[
\frac{c \Delta t}{\Delta x} \leq 1
\]

  – If implicit in time: unconditionally stable for all $\Delta t, \Delta x$
One-Dimensional Example II
Linear Convection (Sommerfeld) Eqn: **4th order approx.**

- **1D exact integral equation still**
  \[
  \frac{d}{dt} \left( \Delta x \Phi_j \right) + f_{j+1/2} - f_{j-1/2} = 0
  \]

- **Use 4th order accurate surface/volume integrals**
  
  - Replace piecewise-constant approx. to \( \varphi(x) \) with **piece-wise quadratic** approx \( (\zeta = x - x_j) \):
    \[
    \varphi(\zeta) = a\zeta^2 + b\zeta + c
    \]
    (note \( \varphi \) defined over more than 1 cell)
  
  - Satisfy \( \Phi_p \)'s (average) constraints, i.e. choose \( a, b, c \) so that:
    \[
    \frac{1}{\Delta x} \int_{-3\Delta x/2}^{-\Delta x/2} \varphi(\zeta) \, d\zeta = \Phi_{j-1}, \quad \frac{1}{\Delta x} \int_{-\Delta x/2}^{+\Delta x/2} \varphi(\zeta) \, d\zeta = \Phi_j, \quad \frac{1}{\Delta x} \int_{\Delta x/2}^{3\Delta x/2} \varphi(\zeta) \, d\zeta = \Phi_{j+1}
    \]
  
    - This gives:
      \[
      a = \frac{\Phi_{j+1} - 2\Phi_j + \Phi_{j-1}}{2\Delta x^2}, \quad b = \frac{\Phi_{j+1} - \Phi_{j-1}}{2\Delta x}, \quad c = \frac{-\Phi_{j-1} + 26\Phi_j - \Phi_{j+1}}{24}
      \]
  
  - Next, we need to evaluate the values of \( \varphi(x) \) at the boundaries so as to compute the advective fluxes at these boundaries: \( f_{j-1/2}^L, f_{j-1/2}^R, f_{j+1/2}^L, f_{j+1/2}^R \)
One-Dimensional Example II
Linear Convection (Sommerfeld) Eqn: 4th order approx.

• Since \( f = c\phi \Rightarrow \text{compute } \phi \text{ at edges:} \)

\[
\phi_{j-1/2}^L = \frac{2\bar{\phi}_j + 5\bar{\phi}_{j-1} - \bar{\phi}_{j-2}}{6}, \quad \phi_{j+1/2}^L = \frac{2\bar{\phi}_{j+1} + 5\bar{\phi}_j - \bar{\phi}_{j-1}}{6},
\]

\[
\phi_{j-1/2}^R = \frac{-\bar{\phi}_{j+1} + 5\bar{\phi}_j + 2\bar{\phi}_{j-1}}{6}, \quad \phi_{j+1/2}^R = \frac{-\bar{\phi}_{j+2} + 5\bar{\phi}_{j+1} + 2\bar{\phi}_j}{6}
\]

• Resolve flux discontinuity \( \Rightarrow \) again, use average values

\[
\hat{f}_{j-1/2} = \frac{\phi_{j-1/2}^L + \phi_{j-1/2}^R}{2} = \frac{c\phi_{j-1/2}^L + c\phi_{j-1/2}^R}{2}
\]

\[
\Rightarrow \hat{f}_{j-1/2} = c\frac{-\bar{\phi}_{j+1} + 7\bar{\phi}_j + 7\bar{\phi}_{j-1} - \bar{\phi}_{j-2}}{12}
\]

\[
\hat{f}_{j+1/2} = \frac{\phi_{j+1/2}^L + \phi_{j+1/2}^R}{2} = \frac{c\phi_{j+1/2}^L + c\phi_{j+1/2}^R}{2}
\]

\[
\Rightarrow \hat{f}_{j+1/2} = c\frac{-\bar{\phi}_{j+2} + 7\bar{\phi}_{j+1} + 7\bar{\phi}_j - \bar{\phi}_{j-1}}{12}
\]

• Done with “integrals” \( \Rightarrow \) we can substitute in 1D conv. eqn:

\[
\frac{d}{dt}(\Delta x \bar{\Phi}_j) + f_{j+1/2} - f_{j-1/2} \approx \frac{d}{dt}(\Delta x \bar{\phi}_j) + \hat{f}_{j+1/2} - \hat{f}_{j-1/2} \Rightarrow \Delta x \frac{d\bar{\phi}_j}{dt} + c\frac{-\bar{\phi}_{j+2} + 8\bar{\phi}_{j+1} - 8\bar{\phi}_{j-1} + \bar{\phi}_{j-2}}{12} = 0
\]

• For periodic domains:

\[
\frac{d}{dt} \bar{\Phi} + \frac{c}{12\Delta x} B_P(-1,-8,0,8,1) \bar{\Phi} = 0
\]
FIGURE 23.3
Centered finite-divided-difference formulas: two versions are presented for each derivative. The latter version incorporates more terms of the Taylor series expansion and is, consequently, more accurate.

Centered Differences

First Derivative
\[
f'(x) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}
\]
\[
f'(x) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}
\]

Second Derivative
\[
f''(x) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}
\]
\[
f''(x) = \frac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2}
\]

Third Derivative
\[
f'''(x) = \frac{f(x_{i+1}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2h^3}
\]
\[
f'''(x) = \frac{-f(x_{i+3}) + 8f(x_{i+2}) - 13f(x_{i+1}) + 13f(x_{i-1}) - 8f(x_{i-2}) + f(x_{i-3})}{6h^3}
\]

Fourth Derivative
\[
f''''(x) = \frac{f(x_{i+2}) - 4f(x_{i+1}) + 6f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{h^4}
\]
\[
f''''(x) = \frac{-f(x_{i+3}) + 12f(x_{i+2}) - 39f(x_{i+1}) + 56f(x_i) - 39f(x_{i-1}) + 12f(x_{i-2}) + f(x_{i-3})}{6h^4}
\]

Error
\[O(h^2)\]
\[O(h^4)\]
One-Dimensional Example III

2nd order approx. of diffusion equation:

\[ \frac{\partial \phi(x,t)}{\partial t} = \nu \frac{\partial^2 \phi(x,t)}{\partial x^2} \]

• 1D exact integral equation same form!

\[ \frac{d(\Delta x \bar{\Phi}_j)}{dt} + f_{j+1/2} - f_{j-1/2} = 0 \]

but with: \( f = -\nu \nabla \phi = -\nu \frac{\partial \phi}{\partial x} \)

• Approximation of surface (flux) integral: Approach 1

– Direct: we know that to second-order (from CDS and from \( \bar{\phi}_j = \phi_j + O(\Delta x^2) \))

\[ f_{j+1/2} = -\nu \frac{\partial \phi}{\partial x} \bigg|_{j+1/2} = -\nu \frac{\bar{\phi}_{j+1} - \bar{\phi}_j}{\Delta x} + O(\Delta x^2) \]

\[ \Rightarrow \hat{f}_{j+1/2} = -\nu \frac{\bar{\phi}_{j+1} - \bar{\phi}_j}{\Delta x} \quad \text{and} \quad \hat{f}_{j-1/2} = -\nu \frac{\bar{\phi}_j - \bar{\phi}_{j-1}}{\Delta x} \]

– Substitute into integral equation:

\[ \frac{d(\Delta x \bar{\phi}_j)}{dt} + \hat{f}_{j+1/2} - \hat{f}_{j-1/2} = \Delta x \frac{d \bar{\phi}_j}{dt} + \nu \frac{\bar{\phi}_{j-1} - 2\bar{\phi}_j + \bar{\phi}_{j+1}}{\Delta x} = 0 \]

– In the matrix form, with Dirichlet BCs:

• Semi-discrete FV scheme is as CDS in space, but in terms of cell-averaged data

\[ \frac{d \bar{\Phi}}{dt} = \frac{\nu}{\Delta x^2} \mathbf{B}(1,-2,1) \bar{\Phi} + (\mathbf{bc}) \]
One-Dimensional Example III

2nd order approx. of diffusion equation:

\[ \frac{\partial \phi(x,t)}{\partial t} = \nu \frac{\partial^2 \phi(x,t)}{\partial x^2} \]

• Approximation of surface (flux) integral: Approach 2

  – Use a piece-wise quadratic approx.: \( \phi(\xi) = a\xi^2 + b\xi + c \) \( \Rightarrow \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \xi} = 2a\xi + b \)

  • Note that \( a, b, c \) remain as before, they are set by the volume average constraints

  • Since \( a, b \) are “symmetric”:

    \[
    f^R_{j+1/2} = f^L_{j+1/2} = -\nu \frac{\partial \phi}{\partial x} \bigg|_{j+1/2} = -\nu \frac{\phi_{j+1} - \phi_j}{\Delta x} + O(\Delta x^2)
    \]

    \[
    f^R_{j-1/2} = f^L_{j-1/2} = -\nu \frac{\partial \phi}{\partial x} \bigg|_{j-1/2} = -\nu \frac{\phi_j - \phi_{j-1}}{\Delta x} + O(\Delta x^2)
    \]

  • There are no flux discontinuities in this case

  – Substitute into integral equation:

    \[
    \frac{d}{dt} \left( \Delta x \phi_j \right) + f_{j+1/2} - f_{j-1/2} = \Delta x \frac{d \phi_j}{dt} + \nu \frac{\phi_{j-1} - 2\phi_j + \phi_{j+1}}{\Delta x} = 0
    \]

  – In the matrix form, with Dirichlet BCs:

    • Semi-discrete FV scheme is as CDS in space, but in terms of cell-averaged data

    \[
    \frac{d}{dt} \Phi = \frac{\nu}{\Delta x^2} \mathbf{B}(1,-2,1) \Phi + (bc)
    \]
Expressing fluxes at the surface based on cell-averaged (nodal) values: Summary of Two Approaches and Boundary Conditions

- **Set-up of surface/volume integrals: 2 approaches (do things in opposite order)**
  1. (i) Evaluate integrals using classic rules (symbolic evaluation); (ii) Then, to obtain the unknown symbolic values, interpolate based on cell-averaged (nodal) values

\[
(i) \quad F_e = \int_{S_e} f_{\phi} \, dA \quad \Rightarrow \quad F_e = G(\phi_e) \\
(ii) \quad \phi_e = H(\phi_p)'s) \equiv H(\phi_p)'s) 
\]

\[
\Rightarrow F_e = F(\phi_p)'s) 
\]

Similar for other integrals:

\[
(S_\phi = \int_{V} s_\phi \, dV, \quad \Phi = \frac{1}{V} \int_{V} \rho \phi dV, \text{ etc})
\]

2. (i) Select shape of solution within CV (piecewise approximation); (ii) impose volume constraints to express coefficients in terms of nodal values; and (iii) then integrate. (this approach was used in the examples).

\[
(i) \quad \phi_{a_i}(x) \equiv J_{a_i}(x) \\
(ii) \quad \int_{V_p} \phi_{a_i}(x) \equiv \phi_{p} 
\]

\[
\Rightarrow \phi_{a_i}(x) \equiv \phi_{p}(x) \\
\Rightarrow F_e = F(\phi_p)'s) 
\]

Similar for higher dimensions:

\[
\phi(x, y) \equiv J_{a_i}(x, y); \quad \text{etc} \\
\phi_{a_i}(x_p, y_p) \equiv \phi_p; \quad \text{etc}
\]

- **Boundary conditions:**
  - Directly imposed for convective fluxes
  - One-sided differences for diffusive fluxes
Approach 1: Evaluate integrals symbolically, then interpolate based on neighboring cell-averages

- Surface/Volume integrals: Approach 1
  (i) Evaluate integrals based on classic rules (symbolic evaluation)
  (ii) Then, to obtain the unknown symbolic values, interpolate based on neighboring cell-averaged (nodal) values

- If we utilize this approach 1
  
  - Symbolic evaluation:
    
    - To evaluate total surface fluxes (convective + diffusive),
      \[ \int_S \vec{F}_\phi \cdot \vec{n} \, dA = \int_S \rho \phi \, (\vec{v} \cdot \vec{n}) \, dA + \int_S \vec{q}_\phi \cdot \vec{n} \, dA \]
      values of \( \phi \) and its gradient normal to the cell face at one or more locations on that face are needed. They have to be expressed as a function of nodal values \( \phi \)
    
    - Similar for volume integrals
  
  - Next is interpolation:
    
    - Express the \( \phi \)’s as a function of nodal values. Numerous possibilities. We already saw some of the most common, provided again next.
Approx. of Surface/Volume Integrals: Classic symbolic formulas

• Surface Integrals \( F_e = \int_{S_e} f_\phi \, dA \)

  - 2D problems (1D surface integrals)
    • Midpoint rule (2\textsuperscript{nd} order): \( F_e = \int_{S_e} f_\phi \, dA = \bar{f}_eS_e = f_eS_e + O(\Delta y^2) \approx f_eS_e \)
    • Trapezoid rule (2\textsuperscript{nd} order): \( F_e = \int_{S_e} f_\phi \, dA \approx S_e \frac{(f_{ne} + f_{se})}{2} + O(\Delta y^2) \)
    • Simpson’s rule (4\textsuperscript{th} order): \( F_e = \int_{S_e} f_\phi \, dA \approx S_e \frac{(f_{ne} + 4f_e + f_{se})}{6} + O(\Delta y^4) \)

  - 3D problems (2D surface integrals)
    • Midpoint rule (2\textsuperscript{nd} order): \( F_e = \int_{S_e} f_\phi \, dA \approx S_e f_e + O(\Delta y^2, \Delta z^2) \)
    • Higher order more complicated to implement in 3D

• Volume Integrals: \( S_\phi = \int_V s_\phi \, dV \), \( \Phi = \frac{1}{V} \int_V \rho \phi dV \)

  - 2D/3D problems, Midpoint rule (2\textsuperscript{nd} order): \( S_p = \int_V s_\phi \, dV = \bar{s}_p \, V \approx s_p \, V \)

  - 2D, bi-quadratic (4\textsuperscript{th} order, Cartesian): \( S_p = \frac{\Delta x \, \Delta y}{36} [16s_p + 4s_s + 4s_n + 4s_w + 4s_e + s_{se} + s_{sw} + s_{ne} + s_{nw}] \)
Interpolations and Differentiations
(to obtain fluxes \( F_e \) as a function of cell-average values)

- **Upwind Interpolation (UDS)** for convective fluxes
  - Approximates \( \phi_e \) by its value at the node upstream of “e”. This is equivalent to using backward or forward-difference approx for a first derivative (depends on direction of flow) => Upwind Differencing Scheme, which is also called Donor-cell.
    
    \[
    \phi_e = \begin{cases} 
    \phi_p & \text{if } (\vec{v} \cdot \vec{n})_e > 0 \\
    \phi_E & \text{if } (\vec{v} \cdot \vec{n})_e < 0 
    \end{cases}
    \]
    
  - This approximation never yields oscillatory solutions (boundedness criterion), but it is **numerically diffusive**:

  - Taylor expansion about \( x_p \): \( \phi_e = \phi_p + (x_e - x_p) \frac{\partial \phi}{\partial x} \bigg|_p + \frac{(x_e - x_p)^2}{2} \frac{\partial^2 \phi}{\partial x^2} \bigg|_p + R_2 \)
  - UDS retains only first term: 1st order scheme in space
    
    \[
    f_e = \rho \phi_e (\vec{v} \cdot \vec{n})_e \approx \hat{f}_e = \rho \phi_p (\vec{v} \cdot \vec{n})_e \quad \Rightarrow \quad \tau_{\Delta x} = \rho (\vec{v} \cdot \vec{n})_e \Delta x \frac{\partial \phi}{\partial x} \bigg|_p + ...
    \]
  - Leading truncation error is “diffusive”, it has the form of a diffusive flux
  - The numerical diffusion is \( \rho (\vec{v} \cdot \vec{n})_e \Delta x \) (has 2 components when flow is oblique to the grid)
Interpolations and Differentiations
(to obtain fluxes “$F_e$” as a function of cell-average values)

- Linear Interpolation (CDS) for convective fluxes
  - Approximates $\phi_e$ (value at face center) by its linear interpolation between two nearest nodes:
    \[ \phi_e = \phi_E \lambda_e + \phi_P (1 - \lambda_e) \quad \text{where} \quad \lambda_e = \frac{x_e - x_P}{x_E - x_P} \]
  - $\lambda_e$ is the interpolation factor
  - This approx. is 2\textsuperscript{nd} order accurate (for convective fluxes):
    - Use Taylor exp. of $\phi_E$ about $x_P$ to eliminate 1\textsuperscript{st} derivative in Taylor exp. of $\phi_e$ (previous slide)
      \[ \phi_E = \phi_p + (x_E - x_p) \left. \frac{\partial \phi}{\partial x} \right|_p + \frac{(x_E - x_p)^2}{2} \left. \frac{\partial^2 \phi}{\partial x^2} \right|_p + R_2 \quad \Rightarrow \left. \frac{\partial \phi}{\partial x} \right|_p = \frac{\phi_E - \phi_p}{x_E - x_p} - \frac{(x_E - x_p)^2}{2} \left. \frac{\partial^2 \phi}{\partial x^2} \right|_p - \frac{R_2}{x_E - x_p} \]
      \[ \Rightarrow \phi_e = \phi_p + (x_e - x_p) \left. \frac{\partial \phi}{\partial x} \right|_p + \frac{(x_e - x_p)^2}{2} \left. \frac{\partial^2 \phi}{\partial x^2} \right|_p + R_2 = \phi_E \lambda_e + \phi_P (1 - \lambda_e) - \frac{(x_e - x_p)(x_E - x_e)}{2} \left. \frac{\partial^2 \phi}{\partial x^2} \right|_p + R'_2 \]
    - Truncation error is proportional to square of grid spacing, on uniform/non-uniform grids.
    - As all approximations of order higher than one, this scheme can provide oscillatory solutions
    - Corresponds to central differences, hence its CDS name (gives avg. if uniform grid spacing)
Interpolations and Differentiations
(to obtain fluxes $F_e$ as a function of cell-average values)

- **Linear Interpolation (CDS) for diffusive fluxes**
  
  - Linear profile between two nearest nodes leads to simplest approx. of gradient (diffusive fluxes)
  
  $$\phi = \phi_E \lambda + \phi_P (1 - \lambda) \quad \Rightarrow \quad \frac{\partial \phi}{\partial x_e} \approx \frac{\phi_E - \phi_P}{x_E - x_P}$$

  $$\lambda = \frac{x - x_P}{x_E - x_P}$$

  - Taylor expansions of $\phi$’s around $x_e$, one obtains:

  $$\tau_{Ax} = \frac{\left(x_e - x_p\right)^2 - \left(x_E - x_e\right)^2}{2 \left(x_E - x_p\right)} \frac{\partial^2 \phi}{\partial x^2} \bigg|_e - \frac{\left(x_e - x_p\right)^3 + \left(x_E - x_e\right)^3}{6 \left(x_E - x_p\right)} \frac{\partial^3 \phi}{\partial x^3} \bigg|_e + R_3$$

  - Approximation is $2^{nd}$ order accurate if $e$ is midway between $P$ and $E$ (e.g. uniform grid)

  - When the grid is non-uniform, the formal accuracy is $1^{st}$ order, but error reduction when grid is refined is asymptotically $2^{nd}$ order
Interpolations and Differentiations
(to obtain fluxes “$F_e$” as a function of cell-average values)

- Quadratic Upwind Interpolation (QUICK), convective fluxes
  - Approx. by quadratic profile between two nearest nodes.
  - In accord with convection, third point chosen on upstream side:
    - i.e. chose W if flow is from P to E, or EE if flow from E to P.

This gives:

$$
\phi_e = \phi_U + g_1 (\phi_D - \phi_U) + g_2 (\phi_U - \phi_{UU})
$$

where D, U and UU denote the downstream, first upstream and second upstream, respectively

- Coefficients in terms of nodal coordinates:
  $$
g_1 = \frac{(x_e - x_U)(x_e - x_{UU})}{(x_D - x_U)(x_D - x_{UU})} \quad ; \quad g_2 = \frac{(x_e - x_U)(x_D - x_e)}{(x_U - x_{UU})(x_D - x_{UU})}
$$

- Uniform grids: coefficients of $\phi$’s are 3/8 for node D, 6/8 for node U and -1/8 for node UU

- Somewhat more complex scheme than CDS (larger computational molecules by one node in each direction)

- Approximation is 3\textsuperscript{rd} order accurate on both uniform and non-uniform grids. For uniform grids:

$$
\phi_e = \frac{6}{8} \phi_U + \frac{3}{8} \phi_D - \frac{1}{8} \phi_{UU} - \frac{3\Delta x^3}{48} \left. \frac{\partial^3 \phi}{\partial x^3} \right|_{UU} + R_3
$$

- But, when this interpolation scheme is used with midpoint rule for surface integral, becomes 2\textsuperscript{nd} order
Interpolations and Differentiations
(to obtain fluxes “\(F_e = f(\phi_e)\)” as a function of cell-average values)

- **Higher Order Schemes** (for convective/diffusive fluxes)
  - Interpolations of order higher than 3 make sense if integrals are also approximated with higher order formulas
  - In 1D problems, if Simpson’s rule (4\(^{th}\) order error) is used for the integral, a polynomial interpolation of order 3 can be used:
    \[
    \phi(x) = a_0 + a_1x + a_2x^2 + a_3x^3
    \]
    \(\Rightarrow\) 4 unknowns, hence 4 nodal values (W, P, E and EE) needed
    = Symmetric formula for \(\phi_e\): no need for “upwind” as with 0\(^{th}\) or 2\(^{nd}\) order polynomials (donor-cell & QUICK)
  - With \(\phi(x)\), one can insert \(\phi_e = \phi(x_e)\) in symbolic integral formula. For a uniform Cartesian grid:
    - **Convective Fluxes:**
      \[
      \phi_e = \frac{27\phi_P + 27\phi_E - 3\phi_W - 3\phi_{EE}}{48}
      \]
      (similar formulas used for \(\phi\) values at corners)
    - **For Diffusive Fluxes (1\(^{st}\) derivative):**
      \[
      \frac{\partial \phi}{\partial x} \bigg|_e = a_1 + 2a_2x + 3a_3x^2 \quad \Rightarrow \quad \text{for a uniform Cartesian grid:} \quad \frac{\partial \phi}{\partial x} \bigg|_e = \frac{27\phi_E - 27\phi_P + \phi_W - \phi_{EE}}{24 \Delta x}
      \]
    - This FV approximation often called a 4\(^{th}\)-order CDS (linear poly. interpol. was 2\(^{nd}\)-order CDS)
    - Polynomials of higher-degree or of multi-dimensions can be used, as well as cubic splines (to ensure continuity of first two derivatives at the boundaries). This increases the cost.
Interpolations and Differentiations (to obtain fluxes “\( F_e = f(\phi_e) \)” as a function of cell-average values)

• **Compact Higher Order Schemes**
  
  – Polynomial of higher order lead too large computational molecules => use deferred-correction schemes and/or compact (Pade’) schemes
  
  – Ex. 1: obtain the coefficients of \( \phi(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \) by fitting two values and two 1st derivatives at the two nodes on either side of the cell face. With evaluation at \( x_e \):
    
    • 4th order scheme: 
      \[
      \phi_e = \frac{\phi_P + \phi_E + \Delta x \left( \frac{\partial \phi}{\partial x} \bigg|_P - \frac{\partial \phi}{\partial x} \bigg|_E \right)}{2} + O(\Delta x^4)
      \]

    • If we use CDS to approximate derivatives, result retains 4th order:
      
      \[
      \phi_e = \frac{\phi_P + \phi_E}{2} + \frac{\phi_P + \phi_E - \phi_W - \phi_{EE}}{16} + O(\Delta x^4)
      \]

  – Ex. 2: use a parabola, fit the values on either side of the cell face and the derivative on the upstream side (equivalent to the QUICK scheme, 3rd order)
    
    \[
    \phi_e = \frac{3}{4} \phi_U + \frac{1}{4} \phi_D + \frac{\Delta x}{4} \left( \frac{\partial \phi}{\partial x} \bigg|_U \right)
    \]

  – Similar schemes are obtained for derivatives (diffusive fluxes), see Ferziger and Peric (2002)

• **Other Schemes**: more complex and difficult to program
  
  – Large number of approximations used for “convective” fluxes: Linear Upwind Scheme, Skewed Upwind schemes, Hybrid. Blending schemes to eliminate oscillations at higher order.
2.29 Numerical Fluid Mechanics
Spring 2015

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.