Measuring the size of a signal

Often, we would like to describe the size of a signal:

1) How much power is being transmitted?

2) How big is my signal compared to the noise?

3) How well does a control system track commanded reference signals? That is, how large is the error signal.

We would like to have a measure of the size of a signal that

1) Conforms to our notions of "big" and "small" signals.

2) Is easy to compute, and has useful properties.
Possible measures of size

a) size = \int_{-\infty}^{\infty} g(t) \, dt

\begin{align*}
&\text{size} \geq 0!
\end{align*}

b) size = \int_{-\infty}^{\infty} |g(t)| \, dt

\begin{align*}
&\text{size} \geq 0, \text{ but messy}
\end{align*}

c) size = \max_t |g(t)|

\begin{align*}
&\text{size} \geq 0
\end{align*}

d) size = \int_{-\infty}^{\infty} g^2(t) \, dt

\begin{align*}
&\text{size} \geq 0
\end{align*}

Which is best? (b), (c), and (d) are all potentially useful, but (d) turns out to be the most useful:

- Works well analytically
- Corresponds to power or energy.
So define
\[ \| g \|^2 = \int_{-\infty}^{\infty} |g(t)|^2 \, dt \]

\[ \| g \| = \sqrt{\| g \|^2} = "\text{norm" (or "2-norm") \text{ of signal}.)} \]

This norm is a direct analog of the length of a vector:
\[ \| x \|^2 = \sum_i x_i^2 \]
Parseval's Theorem

We have defined the norm (squared) of a signal as

\[ \| g \| ^2 = \int_{-\infty}^{\infty} g^2(t) \, dt \]

Would like to compute this in frequency domain, where control design is done.

Define

\[ h(t) = g(t) * g(-t) \]

\[ = \int_{-\infty}^{\infty} g(\tau) g(\tau - t) \, d\tau \]

So

\[ h(\omega) = \int_{-\infty}^{\infty} g(\tau) g(\omega - \tau) \, d\tau = \| g \| ^2 \]

So just need to compute \( h(\omega) \).

Note that

\[ H(f) = G(f)G(-f) = G(f)G^*(f) \]

Inverse FT \( H(f) \) to find \( h(\omega) \):
\[ h(t) = \int_{-\infty}^{\infty} H(f) e^{j2\pi ft} \, df \]

\[ \Rightarrow h(0) = \int_{-\infty}^{\infty} H(f) \, df \]

\[ = \int_{-\infty}^{\infty} G(f) G^*(f) \, df \]

Therefore,

\[ \| g \|^2 = \int_{-\infty}^{\infty} g^2(t) \, dt \]

\[ = \int_{-\infty}^{\infty} |G(f)|^2 \, df \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 \, d\omega \]

This is Parseval's Theorem —

very important.
Example \( g(t) = e^{-at} \), \( a > 0 \)

\[ \int_{-\infty}^{\infty} g^2(t) \, dt = \int_{0}^{\infty} e^{-2at} \, dt = \frac{1}{2a} \]

\[ G(j\omega) = \frac{1}{j\omega + a} \]

\[ \Rightarrow G(j\omega) G(-j\omega) = \frac{1}{j\omega + a} \cdot \frac{1}{-j\omega + a} = \frac{1}{\omega^2 + a^2} \]

\[ \|G\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} \, d\omega \]

\[ = \frac{1}{2\pi} \left[ \frac{1}{a} \tan^{-1} \left( \frac{\omega}{a} \right) \right]_{-\infty}^{\infty} \]

\[ = \frac{1}{2\pi} \left[ \frac{1}{a} \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right) \right] = \frac{1}{2a} \checkmark \]

As expected, the two methods give the same result.
Note: This idea can be extended to Laplace transforms, where

$$\| g \|^2 = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} G(s) G(-s) \, ds$$

Looks harder, but is easier! To evaluate:

1) Expand $G(s)G(-s)$ as partial fractions:

$$G(s)G(-s) = \frac{-1}{(s+a)(s-a)}$$

$$= \frac{1/2a}{s+a} + \frac{-1/2a}{s-a}$$

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pole in left half plane  pole in right half plane

2) Integral = $\sum$ residues of left half plane poles

"residue" = numerator coefficient of first-order poles,