6.003: Signals and Systems

CT Feedback and Control

October 25, 2011
Mid-term Examination #2

Tomorrow, October 26, 7:30-9:30pm,

No recitations on the day of the exam.

Coverage:
- Lectures 1–12
- Recitations 1–12
- Homeworks 1–7

Homework 7 will not be collected or graded. Solutions are posted.

Closed book: 2 pages of notes ($8\frac{1}{2} \times 11$ inches; front and back).

No calculators, computers, cell phones, music players, or other aids.

Designed as 1-hour exam; two hours to complete.

Old exams and solutions are posted on the 6.003 website.
Feedback and Control

Using feedback to enhance performance.

Examples:

- improve performance of an op amp circuit.
- control position of a motor.
- reduce sensitivity to unwanted parameter variation.
- reduce distortions.
- stabilize unstable systems
  - magnetic levitation
  - inverted pendulum
Feedback and Control

Reducing sensitivity to unwanted parameter variation.

Example: power amplifier

Changes in $F_0$ (due to changes in temperature, for example) lead to undesired changes in sound level.
Feedback and Control

Feedback can be used to compensate for parameter variation.

\[ H(s) = \frac{KF_0}{1 + \beta KF_0} \]

If \( K \) is made large, so that \( \beta KF_0 \gg 1 \), then

\[ H(s) \approx \frac{1}{\beta} \]

independent of \( K \) or \( F_0 \)!
Feedback reduces the change in gain due to change in $F_0$. 

$F_0$ (no feedback) 

$\frac{100F_0}{1 + \frac{100F_0}{10}}$ (feedback) 

Gain to Speaker 

$8 < F_0 < 12$ 

MP3 player $X$ $+$ $F_0$ $Y$ 

$8 < F_0 < 12$ 

$\frac{1}{10}$ 

Gain to Speaker 

$0$ $10$ $20$ 

$0$ $10$ $20$ 

$F_0$
Feedback greatly reduces sensitivity to variations in $K$ or $F_0$.

$$\lim_{K \to \infty} H(s) = \frac{KF_0}{1 + \beta KF_0} \to \frac{1}{\beta}$$

What about variations in $\beta$? Aren’t those important?
Check Yourself

What about variations in $\beta$? Aren’t those important?

The value of $\beta$ is typically determined with resistors, whose values are quite stable (compared to semiconductor devices).
Crossover Distortion

Feedback can compensate for parameter variation even when the variation occurs rapidly.

Example: using transistors to amplify power.
Crossover Distortion

This circuit introduces “crossover distortion.”

For the upper transistor to conduct, $V_i - V_o > V_T$.
For the lower transistor to conduct, $V_i - V_o < -V_T$. 
Crossover Distortion

Crossover distortion changes the shapes of signals.

Example: crossover distortion when the input is $V_i(t) = B \sin(\omega_0 t)$. 

![Diagram of crossover distortion](image)
Feedback can reduce the effects of crossover distortion.
Crossover Distortion

When $K$ is small, feedback has little effect on crossover distortion.
Crossover Distortion

Feedback reduces crossover distortion.

\[ V_i \rightarrow + \rightarrow K \rightarrow \begin{cases} +50V \\ -50V \end{cases} \rightarrow V_o \]

\[ V_o(t) \]

\[ K = 2 \]
Crossover Distortion

Feedback reduces crossover distortion.

\[ V_i \xrightarrow{+} K \xrightarrow{-50V} V_o \]

\[ V_o(t) \]

\[ K = 4 \]
Crossover Distortion

Feedback reduces crossover distortion.

\[ V_i \rightarrow +K \rightarrow \begin{cases} +50V \\ -50V \end{cases} \rightarrow V_o \]

\[ V_o(t) \]

\[ K = 10 \]
Crossover Distortion

Demo

- original
- no feedback
- \( K = 2 \)
- \( K = 4 \)
- \( K = 8 \)
- \( K = 16 \)
- original

\[ V_i \rightarrow + \rightarrow K \rightarrow \rightarrow V_o \]

\[ +50V \]

\[ -50V \]

\[ V_o(t) \]

\[ t \]

J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin
Feedback and Control

Using feedback to enhance performance.

Examples:

- improve performance of an op amp circuit.
- control position of a motor.
- reduce sensitivity to unwanted parameter variation.
- reduce distortions.
- stabilize unstable systems
  - magnetic levitation
  - inverted pendulum
Feedback is useful for controlling **unstable** systems.

Example: Magnetic levitation.

\[ i(t) = i_0 \]

\[ y(t) \]
Control of Unstable Systems

Magnetic levitation is unstable.

\[ i(t) = i_o \]

Equilibrium \((y = 0)\): magnetic force \(f_m(t)\) is equal to the weight \(Mg\).

Increase \(y \rightarrow\) increased force \(\rightarrow\) further increases \(y\).

Decrease \(y \rightarrow\) decreased force \(\rightarrow\) further decreases \(y\).

Positive feedback!
Modeling Magnetic Levitation

The magnet generates a force that depends on the distance $y(t)$.

\[ i(t) = io \]

\[ f_m(t) \]

\[ Mg \]

\[ i(t) = i_0 \]

\[ y(t) \]
Modeling Magnetic Levitation

The net force $f(t) = f_m(t) - Mg$ accelerates the mass.

\[ f(t) = f_m(t) - Mg = Ma = M\ddot{y}(t) \]

The net force $f(t) = f_m(t) - Mg$ accelerates the mass.
Modeling Magnetic Levitation

Represent the magnet as a system: input $y(t)$ and output $f(t)$.

\[ i(t) = i_0 \]

\[ f(t) = f_m(t) - Mg = Ma = M\ddot{y}(t) \]
Modeling Magnetic Levitation

The magnet system is part of a feedback system.

\[ f(t) = f_m(t) - Mg = Ma = M\ddot{y}(t) \]

\[ i(t) = i_0 \]

\[ y(t) \]

\[ f(t) \]

\[ y(t) \]

\[ \dot{y}(t) \]

\[ \frac{1}{M} \]

\[ A \]

\[ A \]

\[ y(t) \]
Modeling Magnetic Levitation

For small distances, force grows approximately linearly with distance.

\[ f(t) = f_m(t) - Mg = Ma = M\ddot{y}(t) \]

\[ i(t) = i_0 \]

\[ K \]

\[ y(t) \]

\[ f(t) \]

\[ y(t) \]

\[ K \]

\[ f(t) \]

\[ \frac{1}{M} \]

\[ \ddot{y}(t) \]

\[ A \]

\[ A \]

\[ y(t) \]
"Levitation" with a Spring

Relation between force and distance for a spring is opposite in sign.

\[ F = K \left( x(t) - y(t) \right) = M \ddot{y}(t) \]
Block Diagrams

Block diagrams for magnetic levitation and spring/mass are similar.

Spring and mass

\[ F = K \left( x(t) - y(t) \right) = M \ddot{y}(t) \]

Magnetic levitation

\[ F = Ky(t) = M \ddot{y}(t) \]
Check Yourself

How do the poles of these two systems differ?

Spring and mass

\[ F = K \left( x(t) - y(t) \right) = M \ddot{y}(t) \]

Magnetic levitation

\[ F = Ky(t) = M \ddot{y}(t) \]
Check Yourself

How do the poles of the two systems differ?

Spring and mass

\[ F = K \left( x(t) - y(t) \right) = M \ddot{y}(t) \]

\[ \frac{Y}{X} = \frac{K}{s^2 + \frac{K}{M}} \quad \rightarrow \quad s = \pm j \sqrt{\frac{K}{M}} \]

Magnetic levitation

\[ F = Ky(t) = M \ddot{y}(t) \]

\[ s^2 = \frac{K}{M} \quad \rightarrow \quad s = \pm \sqrt{\frac{K}{M}} \]
Magnetic Levitation is Unstable

\[ i(t) = i_0 \]

\[ f_m(t) \]

\[ Mg \]

\[ y(t) \]

\[ \dot{y}(t) \]

\[ \frac{1}{M} \]

\[ A \]

\[ A \]
Magnetic Levitation

We can stabilize this system by adding an additional feedback loop to control $i(t)$.

\[ f(t) \]
\[ y(t) \]
\[ Mg \]

\[ i(t) = 1.1i_0 \]
\[ i(t) = i_0 \]
\[ i(t) = 0.9i_0 \]
Stabilizing Magnetic Levitation

Stabilize magnetic levitation by controlling the magnet current.

\[ i(t) = i_0 \]

\[ f_m(t) \]

\[ Mg \]

\[ \ddot{y}(t) \]

\[ f(t) \]

\[ \frac{1}{M} \]

\[ A \]

\[ y(t) \]
Stabilizing Magnetic Levitation

Stabilize magnetic levitation by controlling the magnet current.

\[ i(t) = i_o \]

\[ f_{m}(t) \]

\[ y(t) \]

\[ Mg \]

\[ f_i(t) \]

\[ -K_2 \]

\[ \frac{1}{M} \]

\[ K \]

\[ f_o(t) \]

\[ A \]

\[ A \]

\[ y(t) \]
Magnetic Levitation

Increasing $K_2$ moves poles toward the origin and then onto $j\omega$ axis.

\[ x(t) \rightarrow + \rightarrow \frac{K - K_2}{M} \rightarrow \ddot{y}(t) \rightarrow A \rightarrow \dot{y}(t) \rightarrow A \rightarrow y(t) \]

$s$-plane

But the poles are still marginally stable.
Magnetic Levitation

Adding a zero makes the poles stable for sufficiently large $K_2$.

\[ \frac{K-K_2}{M} (s + z_0) \]

Try it: Demo [designed by Prof. James Roberge].
Inverted Pendulum

As a final example of stabilizing an unstable system, consider an inverted pendulum.

\[
\frac{ml^2}{I} \frac{d^2 \theta(t)}{dt^2} = mg \quad \text{force}
\]

\[
l \sin \theta(t) \quad \text{distance}
\]

\[
- m \frac{d^2 x(t)}{dt^2} \quad \text{force}
\]

\[
l \cos \theta(t) \quad \text{distance}
\]
Where are the poles of this system?

\[ m x(t) \frac{d^2 x(t)}{dt^2} = \theta(t) \]

\[ \theta(t) = mg \]

\[ x(t) = l \cos \theta(t) \]

\[ m \frac{d^2 \theta(t)}{dt^2} = ml^2 \frac{d^2 \theta(t)}{dt^2} = mgl \sin \theta(t) - m \frac{d^2 x(t)}{dt^2} l \cos \theta(t) \]
Check Yourself: Inverted Pendulum

Where are the poles of this system?

\[
ml^2 \frac{d^2 \theta(t)}{dt^2} = mgl \sin \theta(t) - m \frac{d^2 x(t)}{dt^2} l \cos \theta(t)
\]

\[
ml^2 \frac{d^2 \theta(t)}{dt^2} - mgl \theta(t) = -ml \frac{d^2 x(t)}{dt^2}
\]

\[
H(s) = \frac{\Theta}{X} = \frac{-mls^2}{ml^2 s^2 - mgl} = \frac{-s^2/l}{s^2 - g/l}
\]

poles at \( s = \pm \sqrt{\frac{g}{l}} \)
Inverted Pendulum

This unstable system can be stabilized with feedback.

Try it. Demo. [originally designed by Marcel Gaudreau]
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