Solutions to In-Class Problems Week 11, Wed.

**Problem 1.** Define the function \( f : \mathbb{N} \to \mathbb{N} \) recursively by the rules

\[
\begin{align*}
f(0) &= 1, \\
f(1) &= 6, \\
f(n) &= 2f(n - 1) + 3f(n - 2) + 4 \quad \text{for } n \geq 2.
\end{align*}
\]

(a) Find a closed form for the generating function

\[
G(x) := f(0) + f(1)x + f(2)x^2 + \cdots + f(n)x^n + \cdots.
\]

**Solution.**

\[
\begin{align*}
G(x) &= f(0) + f(1)x + f(2)x^2 + \cdots + f(n)x^n + \cdots \\
2xG(x) &= 2f(0)x + 2f(1)x^2 + \cdots + 2f(n-1)x^n + \cdots \\
3x^2G(x) &= 3f(0)x^2 + \cdots + 3f(n-2)x^n + \cdots \\
4/(1 - x) &= 4 + 4x + 4x^2 + \cdots + 4x^n + \cdots
\end{align*}
\]

Therefore,

\[
\begin{align*}
G(x) &= 2xG(x) + 3x^2G(x) + \frac{4}{1 - x} + (f(0) - 4) + (f(1) - 2f(0) - 4)x \\
&= 2xG(x) + 3x^2G(x) + \frac{4}{1 - x} + (1 - 4) + (6 - 2 - 4)x \\
&= 2xG(x) + 3x^2G(x) + \frac{4}{1 - x} - 3,
\end{align*}
\]

It follows that

\[
G(x)(1 - 2x - 3x^2) = \frac{4}{1 - x} - 3,
\]

and hence

\[
G(x) = \frac{\frac{4}{1 - x} - 3}{(1 - x)(1 + x)(1 - 3x)} = \frac{4}{(1 - x)(1 + x)(1 - 3x)} - \frac{3}{(1 + x)(1 - 3x)}
\]

\[
= \frac{4 - 3(1 - x)}{(1 - x)(1 + x)(1 - 3x)} = \frac{3x + 1}{(1 - x)(1 + x)(1 - 3x)}.
\]

\[\blacksquare\]
(b) Find a closed form for \( f(n) \). Hint: Find numbers \( a, b, c, d, e, g \) such that

\[
G(x) = \frac{a}{1 + dx} + \frac{b}{1 + ex} + \frac{c}{1 + gx}.
\]

**Solution.** From (1) and the method of partial fractions, we conclude that \( d, e, g = -1, 1, -3 \), respectively. So we want \( a, b, c \) such that

\[
\frac{3x + 1}{(1-x)(1+x)(1-3x)} = \frac{a}{1-x} + \frac{b}{1+x} + \frac{c}{1-3x}
\]

where

\[
3x + 1 = a(1+x)(1-3x) + b(1-x)(1-3x) + c(1-x)(1+x).
\]

Setting \( x = 1 \) in (3), we conclude that \( 4 = a \cdot 2 \cdot (-2) \), so

\[
a = -1.
\]

Setting \( x = -1 \) in (3), we conclude that \( 4 - 3 \cdot 2 = b \cdot 2 \cdot 4 \), so

\[
b = -\frac{1}{4}.
\]

Setting \( x = 1/3 \) in (3), we conclude that \( 4 - 3(2/3) = c \cdot (2/3)(4/3) \), so

\[
c = \frac{9}{4}.
\]

So from (1) and (2), we have

\[
G(x) = \frac{-1}{1-x} + \frac{1/4}{1+x} + \frac{9/4}{1-3x}.
\]

Now the coefficient of \( x^n \) in \( a/(1-x) \) is \( a \), the coefficient in \( b/(1+x) \) is \( b(-1)^n \) and the coefficient in \( c/(1-3x) \) is \( c3^n \). For \( n \geq 2 \), the coefficient in \( G(x) \) is the sum of these coefficients. So

\[
f(n) = -1 + \frac{(-1)^n}{4} + \frac{9}{4}3^n = \frac{3^{n+2} + (-1)^n}{4} - 1.
\]

**Appendix**

**Finding a Generating Function for Fibonacci Numbers**

The Fibonacci numbers are defined by:

\[
f_0 := 0
\]

\[
f_1 := 1
\]

\[
f_n := f_{n-1} + f_{n-2} \quad \text{(for } n \geq 2)\]

Let $F$ be the generating function for the Fibonacci numbers, that is,

$$F(x) := f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$

So we need to derive a generating function whose series has coefficients:

$$\langle 0, 1, f_1 + f_0, f_2 + f_1, f_3 + f_2, \ldots \rangle$$

Now we observe that

$$\langle 0, 1, 0, 0, 0, \ldots \rangle \leftrightarrow x$$
$$\langle 0, f_0, f_1, f_2, f_3, \ldots \rangle \leftrightarrow xF(x)$$
$$+ \langle 0, 0, f_0, f_1, f_2, \ldots \rangle \leftrightarrow x^2 F(x)$$
$$\langle 0, 1 + f_0, f_1 + f_0, f_2 + f_1, f_3 + f_2, \ldots \rangle \leftrightarrow x + xF(x) + x^2 F(x)$$

This sequence is almost identical to the right sides of the Fibonacci equations. The one blemish is that the second term is $1 + f_0$ instead of simply 1. But since $f_0 = 0$, the second term is ok.

So we have

$$F(x) = x + xF(x) + x^2 F(x).$$
$$F(x) = \frac{x}{1 - x - x^2}. \quad (4)$$

**Finding a Closed Form for the Coefficients**

Now we expand the righthand side of (4) into partial fractions. To do this, we first factor the denominator

$$1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x)$$

where $\alpha_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\alpha_2 = \frac{1}{2}(1 - \sqrt{5})$ by the quadratic formula. Next, we find $A_1$ and $A_2$ which satisfy:

$$F(x) = \frac{x}{1 - x - x^2} = \frac{A_1}{1 - \alpha_1 x} + \frac{A_2}{1 - \alpha_2 x} \quad (5)$$

Now the coefficient of $x^n$ in $F(x)$ will be $A_1$ times the coefficient of $x^n$ in $1/(1 - \alpha_1 x)$ plus $A_2$ times the coefficient of $x^n$ in $1/(1 - \alpha_2 x)$. The coefficients of these fractions will simply be the terms $\alpha_1^n$ and $\alpha_2^n$ because

$$\frac{1}{1 - \alpha_1 x} = 1 + \alpha_1 x + \alpha_1^2 x^2 + \cdots$$
$$\frac{1}{1 - \alpha_2 x} = 1 + \alpha_2 x + \alpha_2^2 x^2 + \cdots$$

by the formula for geometric series.

So we just need to find find $A_1$ and $A_2$. We do this by plugging values of $x$ into (5) to generate linear equations in $A_1$ and $A_2$. It helps to note that from (5), we have

$$x = A_1(1 - \alpha_2 x) + A_2(1 - \alpha_1 x),$$
so simple values to use are \( x = 0 \) and \( x = 1/\alpha_2 \). We can then find \( A_1 \) and \( A_2 \) by solving the linear equations. This gives:

\[
A_1 = \frac{1}{\alpha_1 - \alpha_2} = \frac{1}{\sqrt{5}} \\
A_2 = -A_1 = -\frac{1}{\sqrt{5}}
\]

Substituting into (5) gives the partial fractions expansion of \( F(x) \):

\[
F(x) = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \alpha_1 x} - \frac{1}{1 - \alpha_2 x} \right).
\]

So we conclude that the coefficient, \( f_n \), of \( x^n \) in the series for \( F(x) \) is

\[
f_n = \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}} = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).
\]