Relations

Binary relation $R$ from $A$ to $B$

\[
\begin{array}{c}
\text{domain} \\
A
\end{array}
\quad
\begin{array}{c}
\text{codomain} \\
B
\end{array}
\]

\[
\begin{array}{c}
a_1 \\
a_2 \\
a_3
\end{array}
\rightarrow
\begin{array}{c}
b_1 \\
b_2 \\
b_3
\end{array}
\]

\[
\text{graph}(R)
\]

Example

Students

Classes

“is taking”

6.042

6.003

6.012

Example

Arithmetic Expressions

values

1+2

Sqrt(9)

50/10 - 3

“evaluates to”

3

5

2

Example

Cities

“direct bus connection”

Boston

Providence

New York

Cities

“direct bus connection”

Boston

Providence

New York
Relation Abstraction

(Binary) Relation:

\[\text{domain} = \text{set } A\]

\[\text{codomain} = \text{set } B\]

\[\text{graph} = \text{subset of } A \times B\]

\[\text{graph}(R) = \{(a_1,b_1), (a_1,b_3), (a_3,b_3)\}\]

\[A \times B = \{(a_1,b_1), (a_1,b_2), (a_1,b_3),
(a_2,b_1), (a_2,b_2), (a_2,b_3),
(a_3,b_1), (a_3,b_2), (a_3,b_3)\}\]

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Types of Binary Relations on A

• Equivalence

• Partial Orders

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Equivalence Relations

• Equivalence (mod 4):
  \[1 \equiv 5\] (same remainder/4)

• Propositional equivalence:
  \[P \land Q \equiv \overline{P} \lor \overline{Q}\] (same truth table)

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Def. of Equivalence on Set \(A\)

There is a function, \(f\), on \(A\) such that

\[a \ R \ b \iff f(a) = f(b)\]
Equivalence Relations

- Equivalence (mod 4):
  \[ 1 \equiv 5 \quad (\text{same remainder}/4) \]
  \[ f(x) = x \mod 4 \]

Hash Functions

How to map a large address space into a smaller address space?

Large set of addresses \( h \rightarrow \) Small address space

So no collisions occur?

\[ h(\langle \text{name1} \rangle) = h(\langle \text{name2} \rangle) \]

Hash Collision Equivalence

\[ h(\langle \text{name1} \rangle) = h(\langle \text{name2} \rangle) \]

Collides with is an equivalence relation (on addresses in large space)

Athena Equivalence

Athena assigns user directories based on the first two letters of a username:

\[ \text{rab} & \text{raej in r/a/} \]

Same User Directory

```
$ls -l athena.dialup.mit.edu
```

- Names with same first 2 letters:
  \[ \text{Ben} \equiv \text{Betty} \]
  \[ f(\text{name}) = \text{first two letters} \]
**Partitions**

*Theorem:* An equivalence relation partitions its domain into collections of equivalent elements called *equivalence classes.*

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**Athena Partition**

- All names starting with "aa"
- All names starting with "ab"
- All names starting with "ac"
- All names starting with "zz"

26 × 26 equivalence classes

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**Some properties of relations:**

Relation $R$ on set $A$ is  

- **Reflexive:** if $aRa$ for all $a \in A$.  
- **Symmetric:** if $aRb \rightarrow bRa$ for all $a,b \in A$.  
- **Transitive:** if $(aRb \land bRc) \rightarrow aRc$ for all $a,b,c \in A$.  

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**Equivalence Relation Properties**

Equivalence Relation $R$ on set $A$ is  

- **Reflexive:** $aRa$  
- **Symmetric:** $aRb \rightarrow bRa$  
- **Transitive:** $[aRb \land bRc] \rightarrow aRc$

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**Equivalence Relation Properties Theorem:**

$R$ is an equivalence relation iff it is  

- Reflexive,  
- Symmetric,  
- Transitive

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**Team Problems**

Problems 1 & 2
Ordering Relations

- \( \leq \) on the Integers
- \( < \) on the Reals
- \( \subseteq \) on Sets (subset)
- \( \subset \) on Sets (proper subset)

Partial Orders

\( y \ll x \) (much less than)

\( y + 2 \leq x \) (say)

\[ \neg [3 \ll 4] \quad \neg [4 \ll 3] \]

incomparable

Partial Orders

The proper subset relation, \( \subset \), on sets is the canonical example.

(Proper) Subset Relation

\{1,2,3,5,10,15,30\}

\{1,3,5,15\}

\{1,3\} \{1,2\}

\{1\}

Partial Order: divides

\( a \) divides \( b \) (\( a \mid b \)) iff \( ka = b \) for some \( k \in \mathbb{N} \)

Partial Order: divides

\( 1 \mid 2 \)

\( 2 \mid 10 \)

\( 5 \mid 10 \)

\( 15 \mid 30 \)
Def. of Partial Order on Set $A$

There is a set-valued function, $g$, on $A$ such that

$$a \ R \ b \iff g(a) \subset g(b)$$

for $a \neq b$

Divides & Subset

same "shape"

Divides & Subset

Let

$$g(n) ::= \text{divisors of } n$$

$$n \mid m \iff g(n) \subset g(m)$$

for $n \neq m$

Subset Relation

Properties of $\subset$

$[A \subset B \text{ and } B \subset C]$ implies $A \subset C$

Transitive

$A \subset B$ implies $\neg (B \subset A)$

for $A \neq B$

Antisymmetric
Axioms for Partial Order

Theorem: \( R \) is a partial order iff
- Transitive & Antisymmetric

(Compare to Equivalence:
- Reflexive, Transitive, Symmetric.)

Total Order on \( A \)

Partial Order, \( R \), such that
\[ aRb \text{ or } bRa \]
for all \( a\neq b \in A \)

Total Orders

\[ a < b \text{ or } b < a \]
(for numbers \( a \neq b \))

Total Orders

\[ a \leq b \text{ or } b \leq a \]
(for all \( a, b \))

Team Problems

Problems 3 & 4