Damped Harmonic Oscillators

In the last session we modeled a spring-mass-dashpot system with the constant coefficient linear DE

\[ m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}, \]

where \( m \) is the mass, \( b \) is the damping constant, \( k \) is the spring constant and \( x(t) \) is the displacement of the mass from its equilibrium position.

We then assumed the external force \( F_{\text{ext}} = 0 \) and used the characteristic equation technique to solve the homogeneous equation

\[ m\ddot{x} + b\dot{x} + kx = 0. \quad (1) \]

**Restrictions on the coefficients:** The algebra does not require any restrictions on \( m, b \) and \( k \) (except \( m \neq 0 \) so that the equation is genuinely second order). But, since this is a physical model, we will now require \( m > 0, b \geq 0 \) and \( k > 0 \).

**The Damped Harmonic Oscillator:** The undamped \((b = 0)\) system has equation

\[ m\ddot{x} + kx = 0. \]

At this point you should have memorized the solution and also be able to solve this equation using the characteristic roots. The solution is

\[ x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \cos(\omega t - \phi). \]

Here \( \omega = \sqrt{k/m} \) and the solution is given in both rectangular and amplitude-phase form. The solution is always a sinusoid, which we consider a simple oscillation, and we call this system a **simple harmonic oscillator**.
When we add damping we call the system in (1) a **damped harmonic oscillator**. This is a much fancier sounding name than the spring-mass-dashpot. It emphasizes an important fact about using differential equations for modeling physical systems. It doesn’t matter whether $x$ measures position or current or some other quantity. Any system modeled by equation (1) will respond just like the spring-mass-dashpot; that is, all damped harmonic oscillators exhibit similar behavior. We will see an important example of this principle when we study the case of an RLC electrical circuit.