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Spring 2008

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Control of Manufacturing Processes

Subject 2.830/6.780/ESD.63

Spring 2008

Lecture #13

Modeling Testing and Fractional Factorial Designs

April 1, 2008

Outline

- Full Factorial Models
  - Contrasts
  - Extension to 2^k
  - Model Term Significance: ANOVA
  - Checking Adequacy of Model Form
    - Tests for higher order fits (curvature)
- Experimental Design
  - Blocks and Confounding
  - Single Replicate Designs
  - Fractional Factorial Designs

NB: Read Montgomery Chapter 12

2^2 Model Based on Contrasts

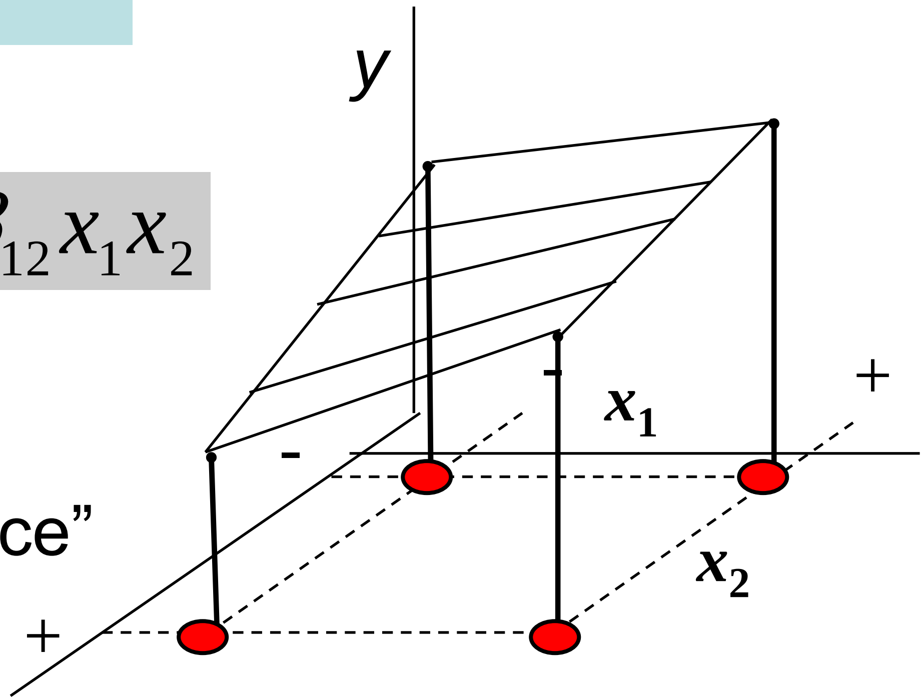
Two factor, two level experiments:

$$\hat{y} = \bar{\bar{y}} + \frac{A}{2}x_1 + \frac{B}{2}x_2 + \frac{AB}{2}x_1 x_2$$

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

(Regression model)

This defines a 3-D "ruled surface"



General Form for Contrasts

Trial	A	B	AB
(1)	-	-	+
a	+	-	-
b	-	+	-
ab	+	+	+

$$A:[a+ab-b-(1)]$$

$$B:[b+ab-a-(1)]$$

$$AB:[ab+(1)-a-b]$$

$\text{Contrast}_A = \text{Trial Column} \cdot A$

$\text{Contrast}_B = \text{Trial Column} \cdot B$

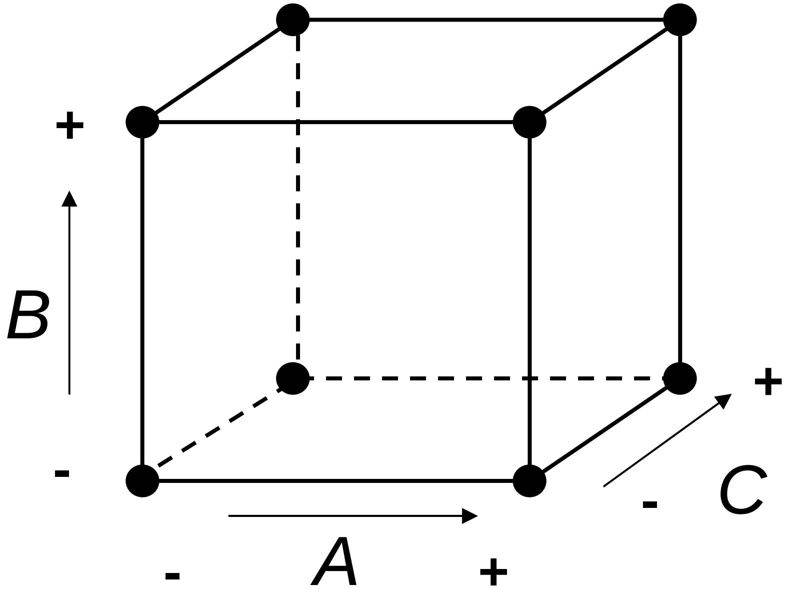
$\text{Contrast}_{AB} = \text{Trial Column} \cdot AB$

Extension to 2^k

Consider 2^3 (3 factors, 2 levels each factor):

Run Number	Treatment Combination		Factor Levels x_1 A	x_2 B	x_3 C
1	(1)	y_1	-1	-1	-1
2	a	y_2	1	-1	-1
3	b	y_3	-1	1	-1
4	ab	y_4	1	1	-1
5	c	y_5	-1	-1	1
6	ac	y_6	1	-1	1
7	bc	y_7	-1	1	1
8	abc	y_8	1	1	1

Generalization



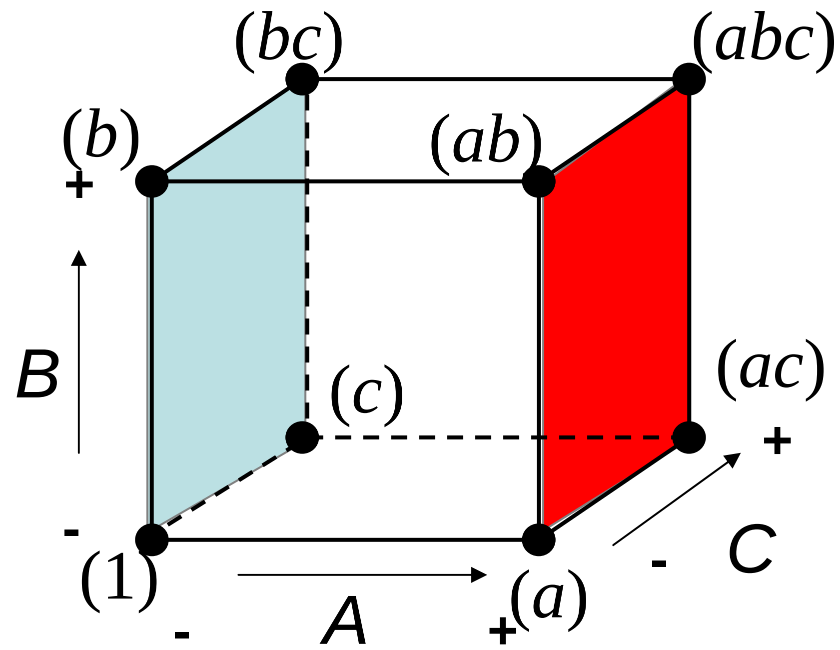
$$2^k$$

number of factors

number of levels

Courtesy of Dan Frey. Used with permission.

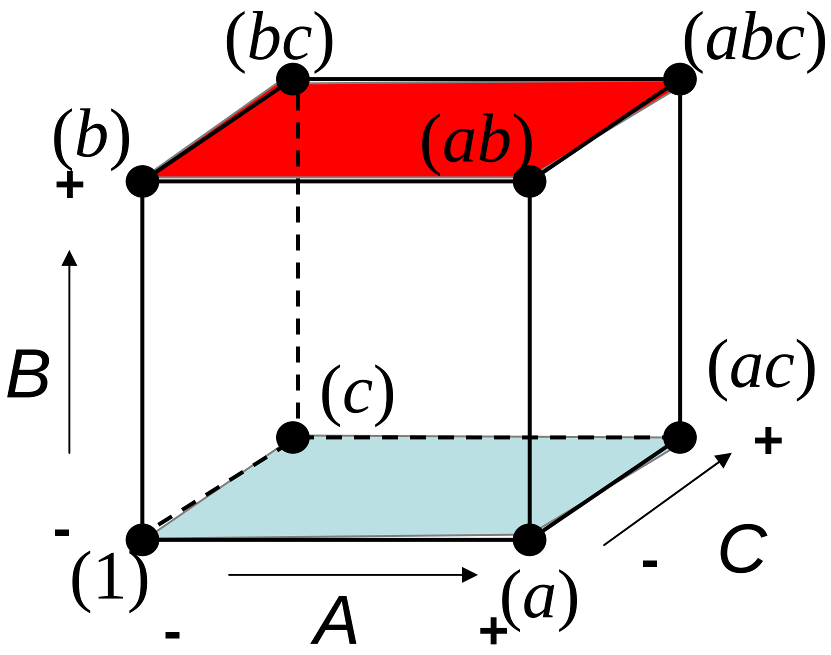
Contrasts as "Surface" Average Differences



$$A = \frac{1}{4}\left[(abc)+(ab)+(ac)+(a)\right] - \frac{1}{4}\left[(b)+(c)+(bc)+(1)\right]$$

Courtesy of Dan Frey. Used with permission.

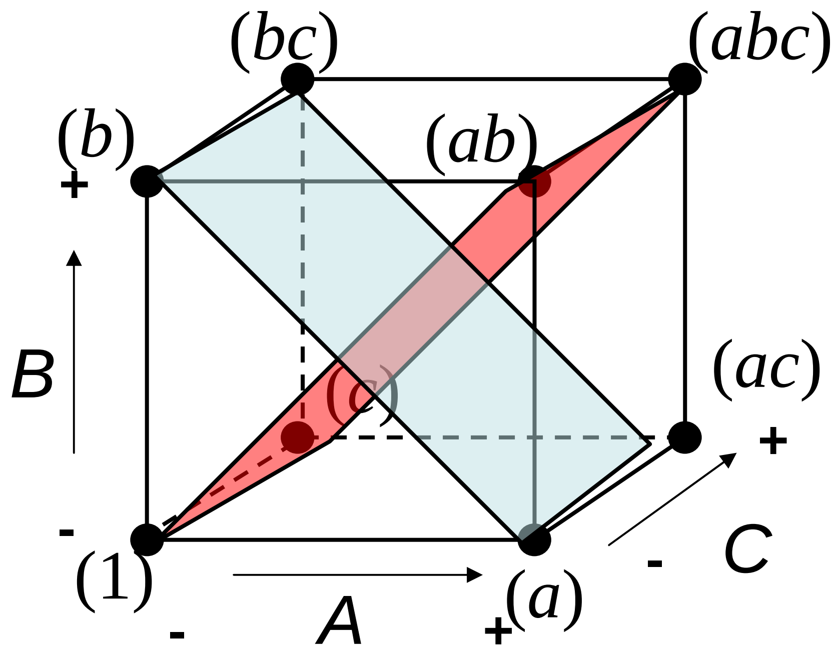
Contrasts for Main Effect



$$B = \frac{1}{4}\left[(abc) + (ab) + (bc) + (b)\right] - \frac{1}{4}\left[(a) + (c) + (ac) + (1)\right]$$

Courtesy of Dan Frey. Used with permission.

Contrasts for Interaction Effect



$$AB = \frac{1}{4}\left[(1)+(ab)+(c)+(abc)\right] - \frac{1}{4}\left[(a)+(b)+(ac)+(bc)\right]$$

Courtesy of Dan Frey. Used with permission.

Contrasts for 2^3

Treament Combination	Factorial Combination							
	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

$$\text{Contrast } A: [a + ab + ac + abc - b - c - bc - (1)]$$

$$\text{Contrast } ABC: [a + b + c + abc - ab - ac - bc - (1)]$$

$$\text{Effect} = \frac{\text{Contrast}}{n2^{k-1}}$$

where n is the number of replicates at each treatment combination

$$\Rightarrow \quad A = \frac{1}{4n}[a + ab + ac + abc - b - c - bc - (1)]$$

Factorial Combinations

Treament Combination	Factorial Combination							
	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

Note: this is the scaled X matrix in the regression model

Relationship to Regression Model

$$\underline{\beta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \underline{y}$$

$\Downarrow$ $\underline{y}$ is data from experimental design $\mathbf{X}$

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$ regression model

- A is the Effect of input 1 averaged over all other input changes (-1 to +1 or a total range of 2)
- B is the Effect of input 2 averaged over all other input changes, …….

$$\beta_0 = \bar{y} \quad \beta_1 = \frac{A}{2}; \quad \beta_2 = \frac{B}{2}; \quad \beta_{12} = \frac{AB}{2}$$

or

$$\hat{y} = \bar{\bar{y}} + \frac{A}{2}x_1 + \frac{B}{2}x_2 + \frac{AB}{2}x_1 x_2$$

ANOVA for 2^k

- Now have more than one "effect"
- We can derive:

$$SS_{Effect} = (Contrast)^2 / n2^k$$

- And it can be shown that:

$$SS_{Total} = SS_A + SS_B + SS_{AB} + SS_{Error}$$

ANOVA Table

Source	SS	d.o.f.	MS	F_0	F_{crit}
A	$\frac{Contrast_A^2}{2^2 n}$	1	SS_A	$\frac{MS_A}{MS_E}$	$F_{1,2n-4,\alpha}$
B	$\frac{Contrast_B^2}{2^2 n}$	1	SS_B	$\frac{MS_B}{MS_E}$	
AB	$\frac{Contrast_{AB}^2}{2^2 n}$	1	SS_C	$\frac{MS_{AB}}{MS_E}$	
Error	SS_E	$(2^2 \cdot n) - 3$	$\frac{SS_E}{(2^2 \cdot n) - 3}$		
Total	$\Sigma\Sigma(y_{ij}-\overline{y})^2$	$(2^2 \cdot n) - 1$			

Alternative Form

Source	SS	d.o.f.	MS	F
mean	$nm\,\beta_0^{\,2}$	1	$\frac{SS(\beta_0)}{1}$	$\frac{MS(\beta_0)}{MS(\varepsilon)}$
x_1	$nm\,\beta_1^{\,2}$	1	$\frac{SS(\beta_1)}{1}$	$\frac{MS(\beta_1)}{MS(\varepsilon)}$
x_2	$nm\,\beta_2^{\,2}$	1	$\frac{SS(\beta_2)}{1}$	$\frac{MS(\beta_2)}{MS(\varepsilon)}$
x_{12}	$nm\,\beta_{12}^{\,2}$	1	$\frac{SS(\beta_{12})}{1}$	$\frac{MS(\beta_{12})}{MS(\varepsilon)}$
ε	$\sum_{i=1}^{m}\sum_{j=1}^{n}\varepsilon_{ij}$	$mn-4$	$\frac{SS(\varepsilon)}{(mn-4)}$	
total	$\sum_{i=1}^{m}\sum_{j=1}^{n}y_{ij}$	mn		

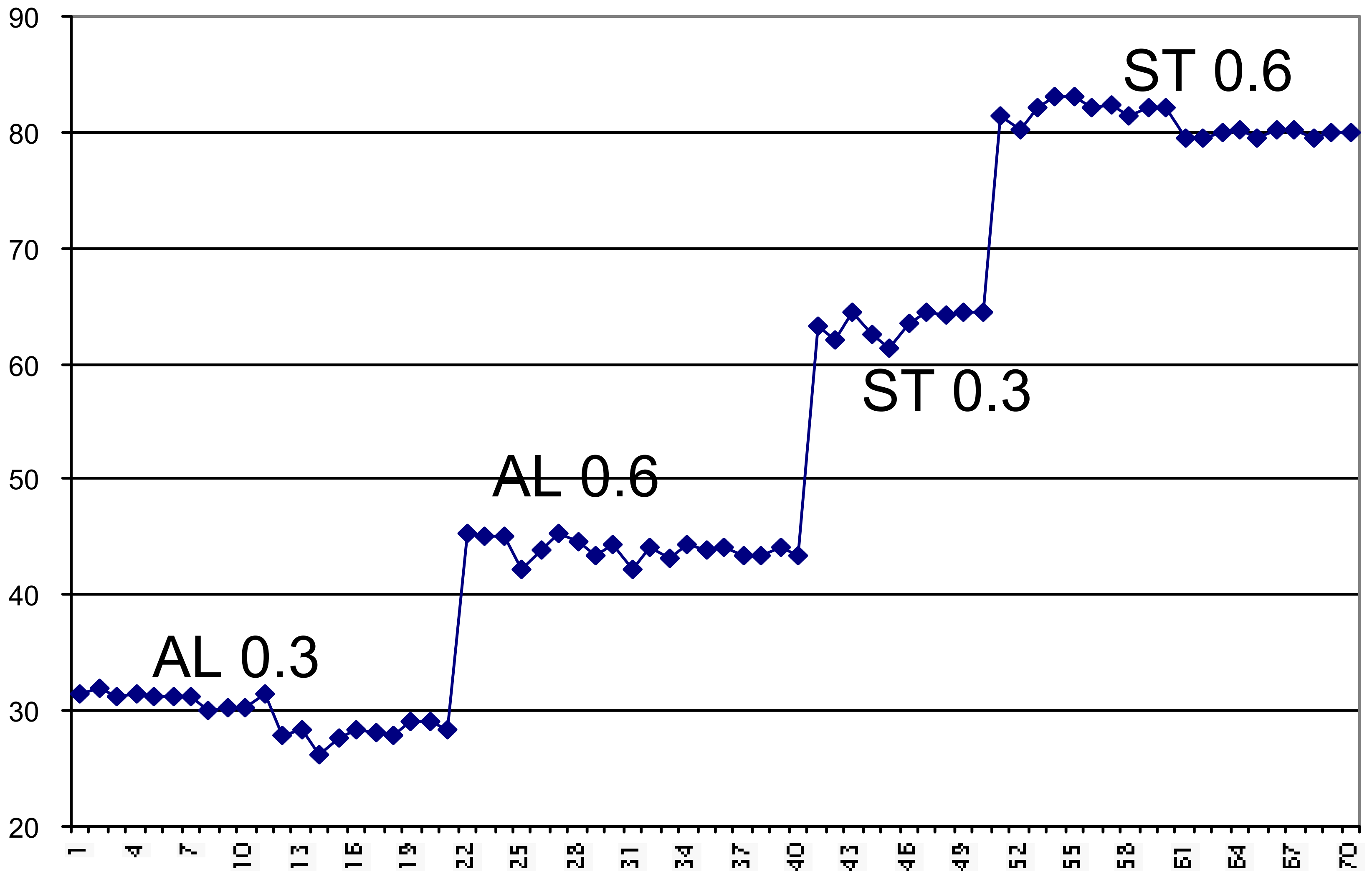
n = replicates

$m = 2^k$

SS_{Total} includes the grand mean in this formulation

For all terms $\quad F_{crit} = F_{1,\,mn-4,(1-\alpha)}$

Recall the Brakeforming Data (MIT 2002)



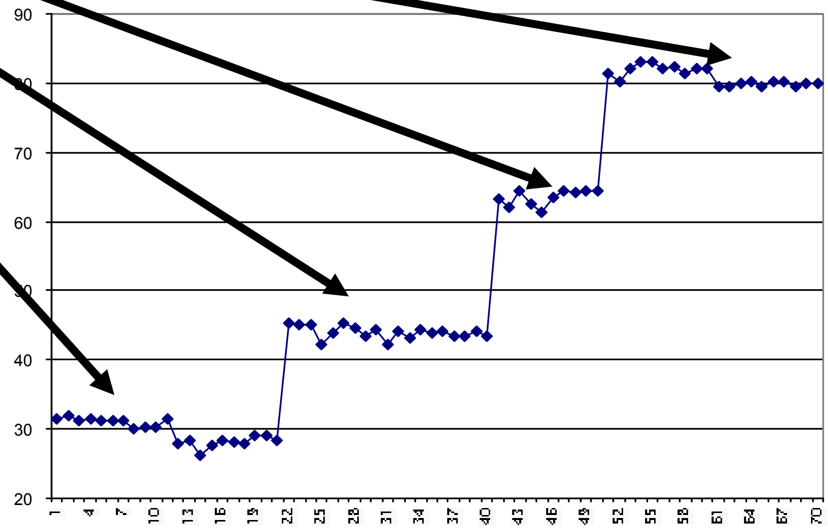
Inputs and Levels

- Inputs
  - Punch Depth (x_1)
    - 0.3 In (-1)
    - 0.6 in (+1)
  - Material Type/Thickness (x_2) (e.g.. bending stiffness)
    - Aluminum (-`1)
    - Steel (+1)
- 2 Inputs 2 levels each - 2^2 Model
- Output: Angle (y)

Data Table for 2^2 Model

Test	x1	x2	yi1	yi2	yi3	yi4	yi5	yi6	yi7	yi8	yi9	yi10
1	-1	-1	31.45	32.00	31.15	31.45	31.15	31.15	31.15	30.15	30.20	30.30
2	-1	1	45.30	45.10	45.00	42.15	44.00	45.35	44.55	43.30	44.30	42.15
3	1	-1	63.15	62.00	64.50	62.55	61.30	63.45	64.40	64.10	64.45	64.35
4	1	1	81.45	80.15	82.20	83.00	83.05	82.20	82.25	81.45	82.15	82.00

- x_1 : Material
- x_2 : Depth
- 4 Tests
- 10 Replicates



Looking only at Mean Response

Test	x1	x2	yibar
1	-1	-1	31.02
2	-1	1	44.12
3	1	-1	63.43
4	1	1	81.99

$\underline{y} =$

31
44.1
63.4
82

$\mathbf{X} =$

1	x1	x2	x1x2
1	-1	-1	1
1	-1	1	-1
1	1	-1	-1
1	1	1	1

Model and Interpretation

- Solving $\underline{\beta}=X^{-1}\,\underline{y}$

$$\underline{\beta}= \begin{bmatrix} 55.1 \\ 17.6 \\ 7.92 \\ 1.36 \end{bmatrix}$$

$$y = 55.1 + 17.6x_1 + 7.9x_2 + 1.4x_1x_2 + \varepsilon$$

Residual Analysis

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + h.o.t. + \varepsilon$$

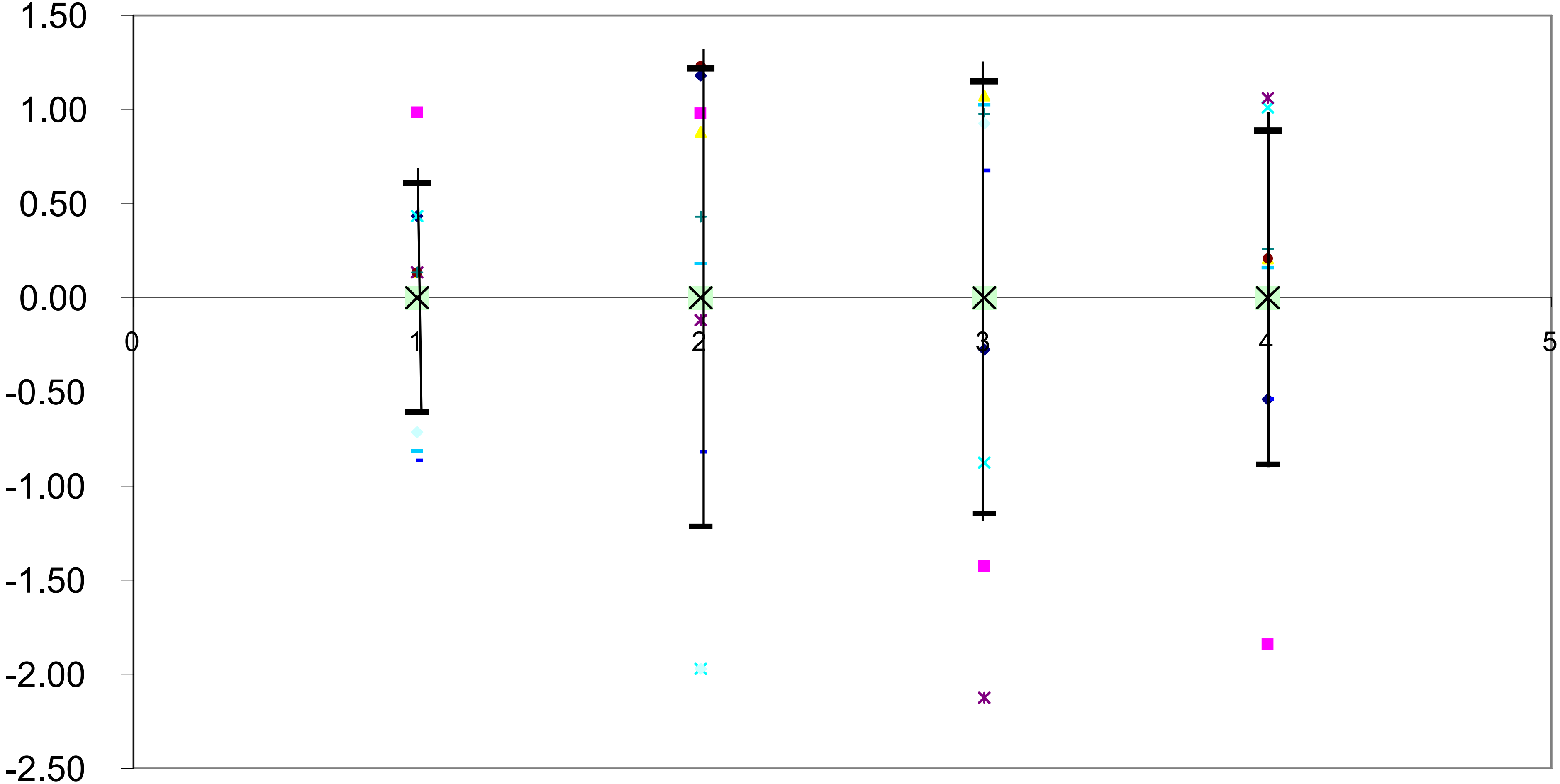
$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2$$

$$y - \hat{y} = h.o.t. + \varepsilon = \text{residual}$$

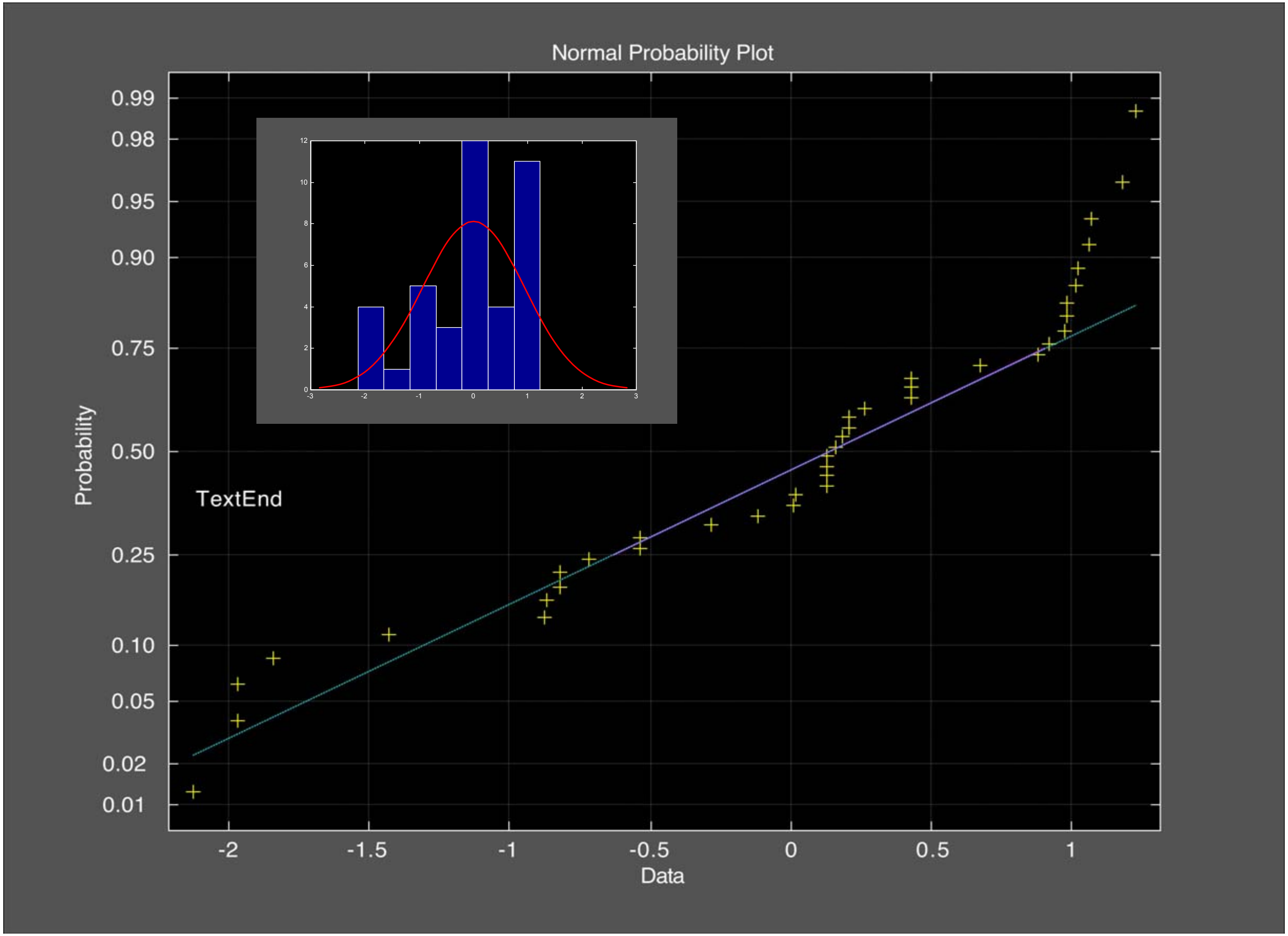
Properties of residual?

- if model is "correct"
- if model of error is $\sim N(0,\sigma^2)$

Residuals (ε) with Test



Residual Distribution



Aside: *Use of All Data*

X η

1	x1	x2	x1x2		y
1	-1	-1	1		31.45
1	-1	1	-1		45.30
1	1	-1	-1		63.15
1	1	1	1		81.45
1	-1	-1	1		32.00
1	-1	1	-1		45.10
1	1	-1	-1		62.00
1	1	1	1		80.15
1	-1	-1	1		31.15
1	-1	1	-1		45.00
1	1	-1	-1		64.50
1	1	1	1		82.20
1	-1	-1	1		31.45
1	-1	1	-1		42.15
1	1	-1	-1		62.55
1	1	1	1		83.00
1	-1	-1	1		31.15
1	-1	1	-1		44.00
1	1	-1	-1		61.30
1	1	1	1		83.05
1	-1	-1	1		31.15
1	-1	1	-1		45.35
1	1	-1	-1		63.45
1	1	1	1		82.20
1	-1	-1	1		31.15
1	-1	1	-1		44.55
1	1	-1	-1		64.40
1	1	1	1		82.25
1	-1	-1	1		30.15
1	-1	1	-1		43.30
1	1	-1	-1		64.10
1	1	1	1		81.45
1	-1	-1	1		30.20
1	-1	1	-1		44.30
1	1	-1	-1		64.45
1	1	1	1		82.15
1	-1	-1	1		30.30
1	-1	1	-1		42.15
1	1	-1	-1		64.35
1	1	1	1		82.00

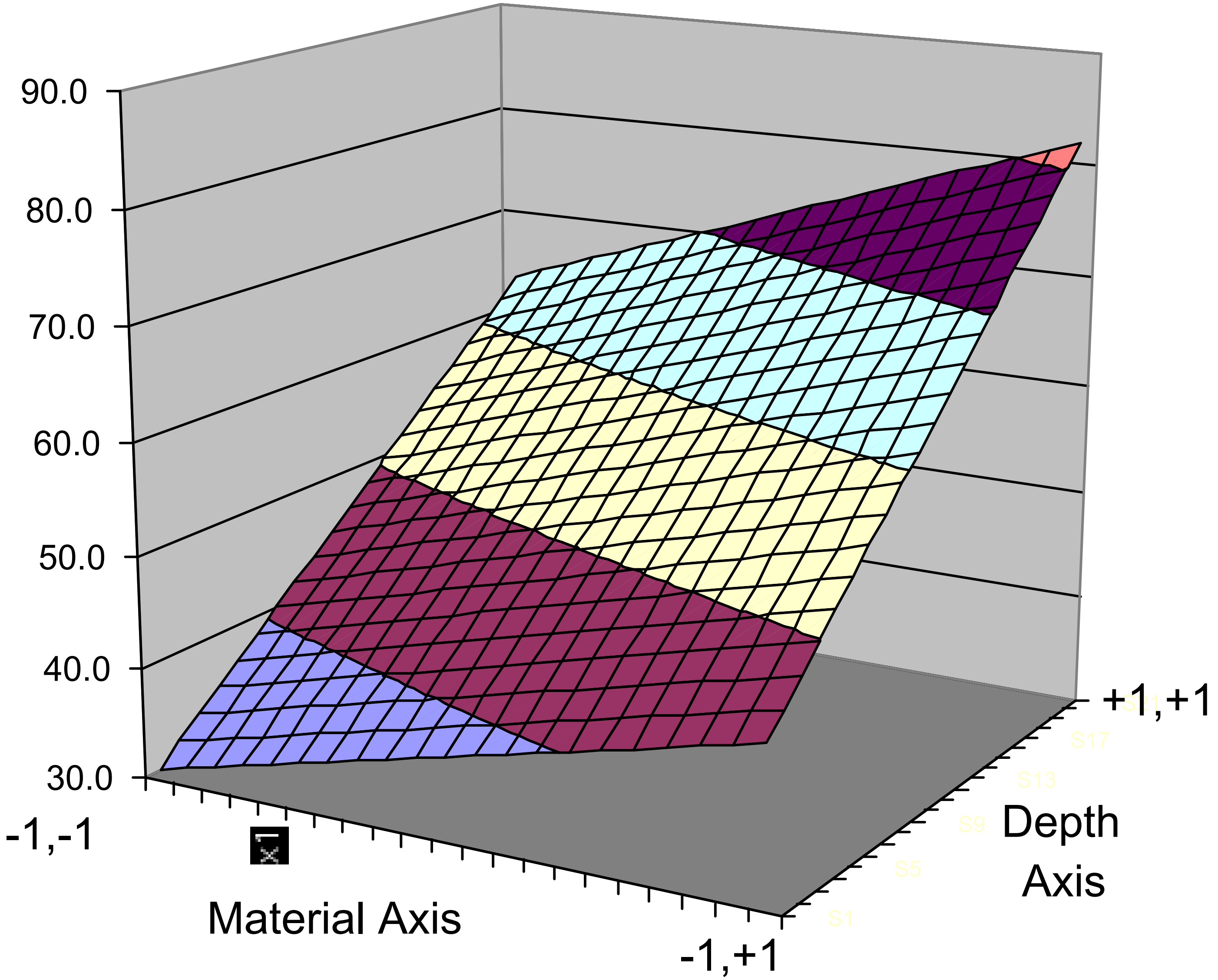
$$\underline{\beta} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\underline{\eta}$$

$$\underline{\beta} = \begin{bmatrix} 55.1 \\ 17.6 \\ 7.92 \\ 1.36 \end{bmatrix}$$

Same as before!

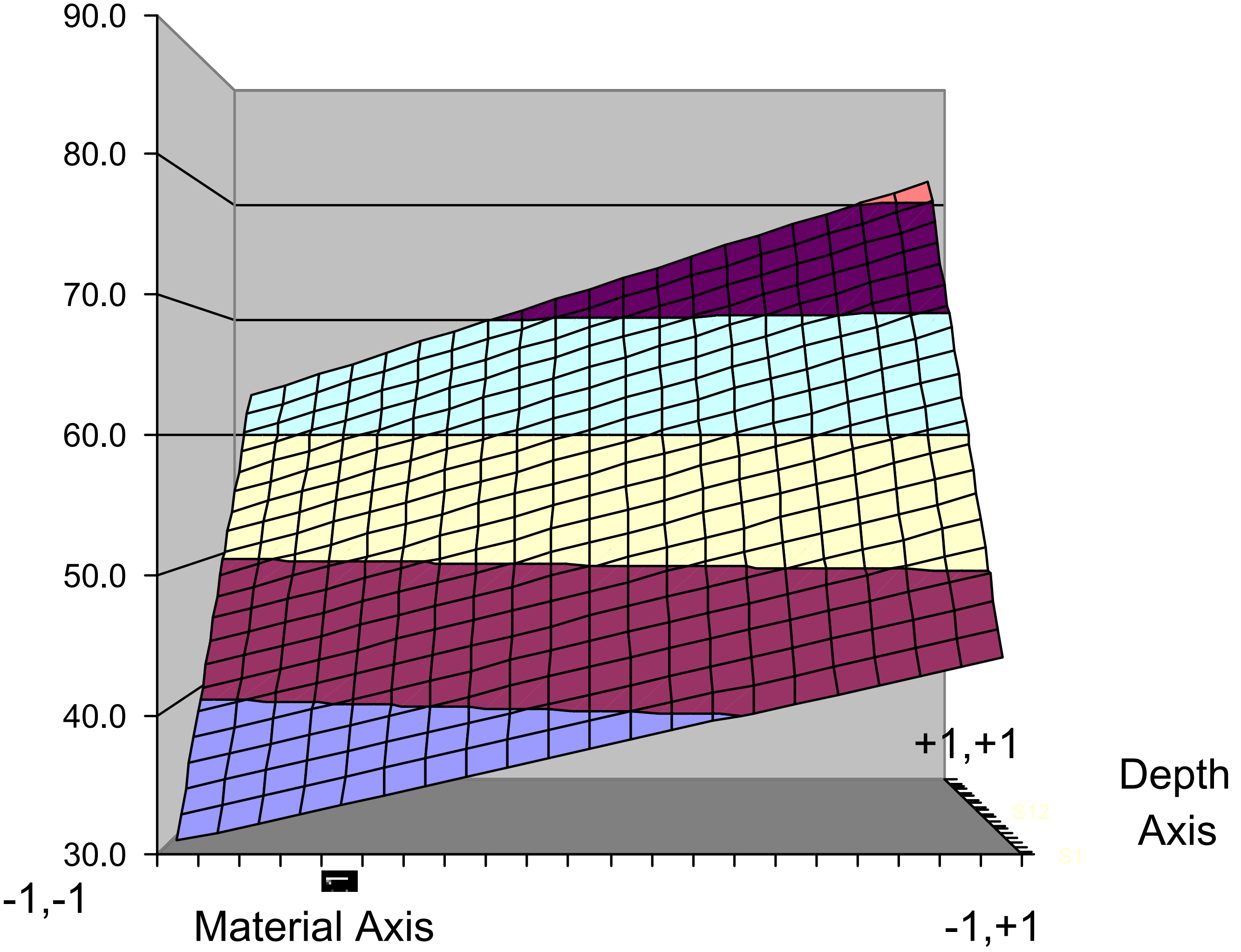
Response Surface

-1 0.3 in
+1 0.6 in
-1 Al
+1 St



Side View of Surface

- -1 0.3 in
- +1 0.6 in
- -1 Al
- +1 St



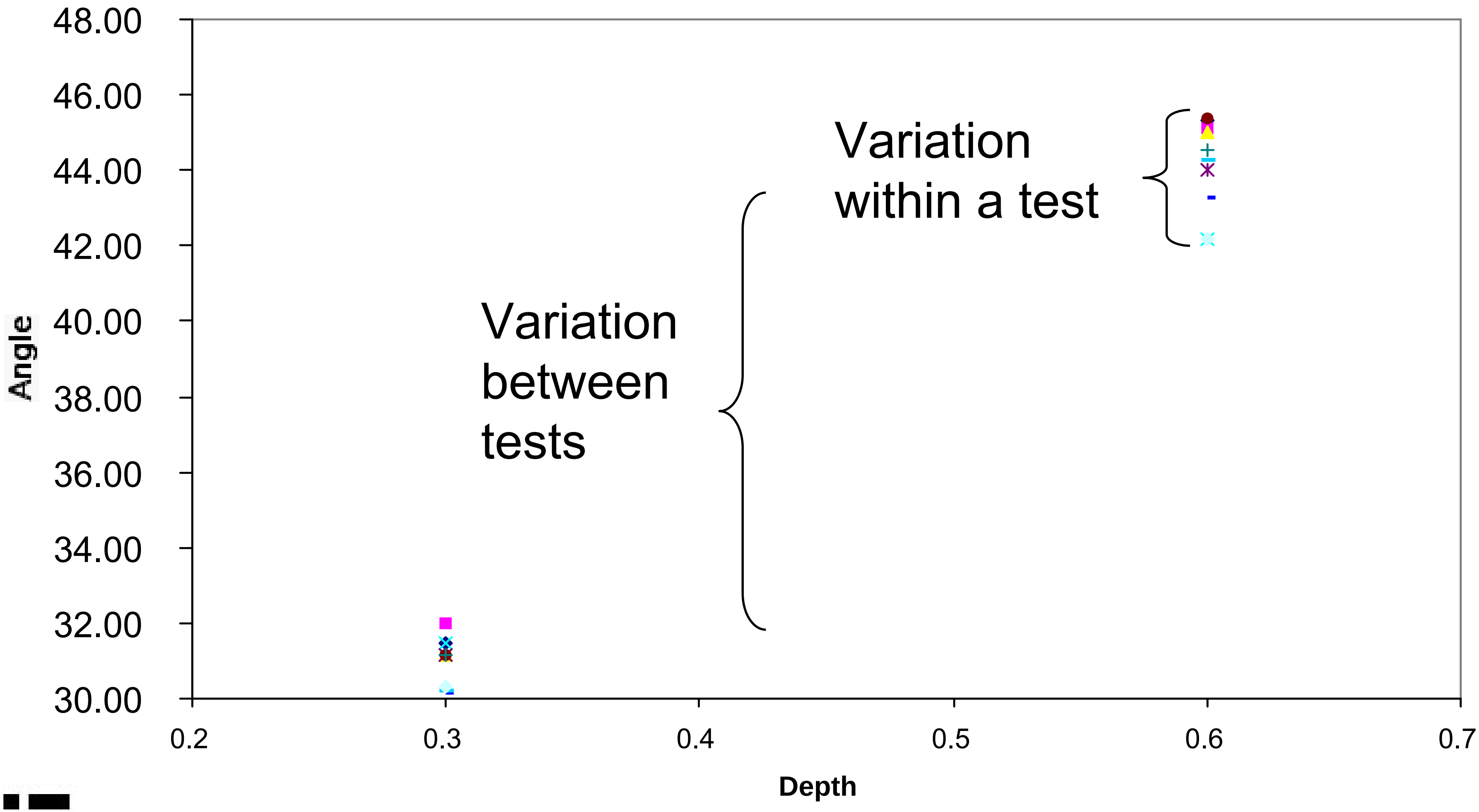
- Degree of interaction?

Are the Model Terms Significant?

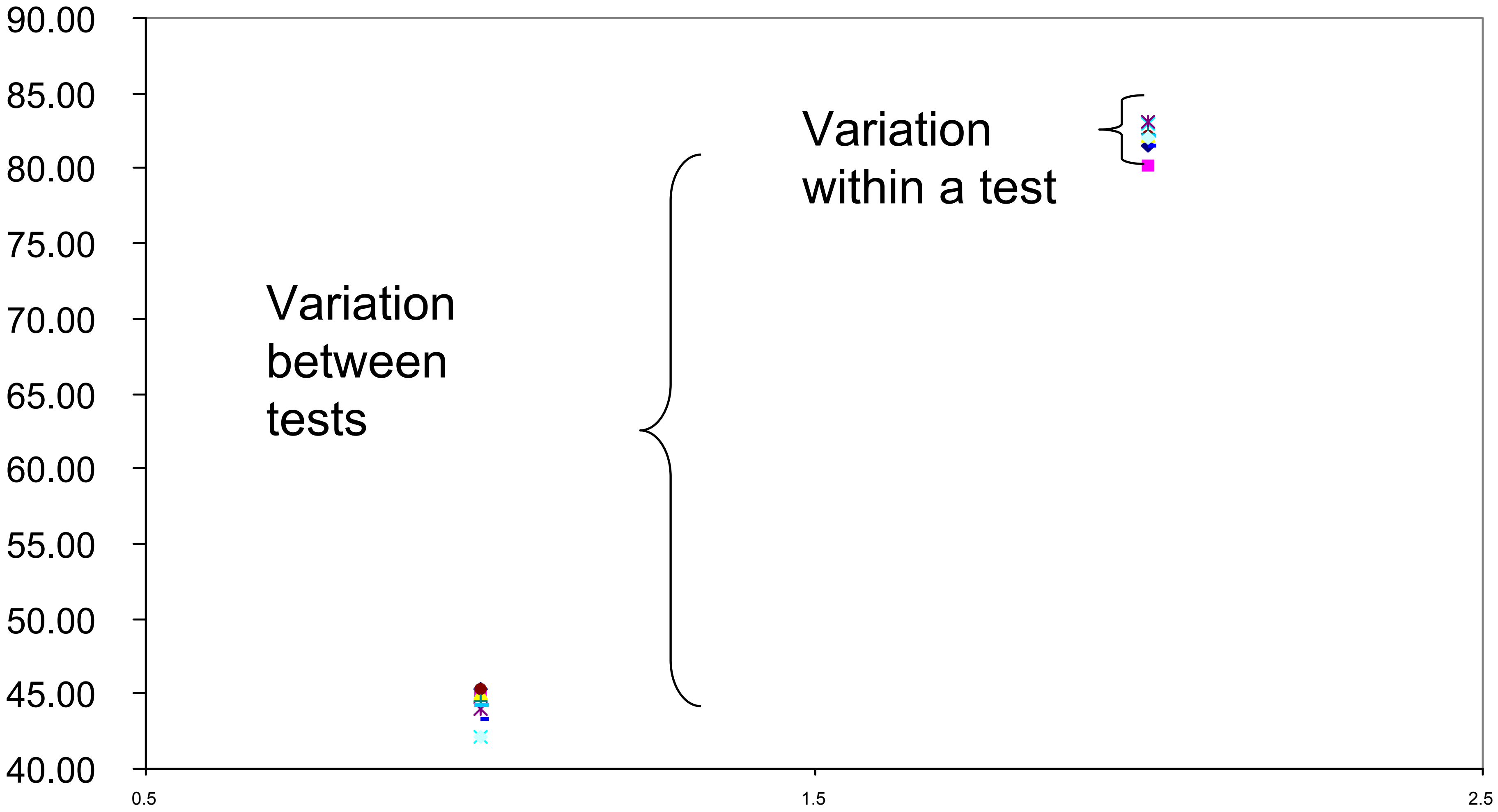
- Mean: β_0
- Effect of Depth: $2\beta_1$
- Effect of Material: $2\beta_2$
  - Contaminated by simultaneous change of modulus, thickness and yield
- Interaction of Depth and Material: $2\beta_{12}$

Look at Single Variable Plots

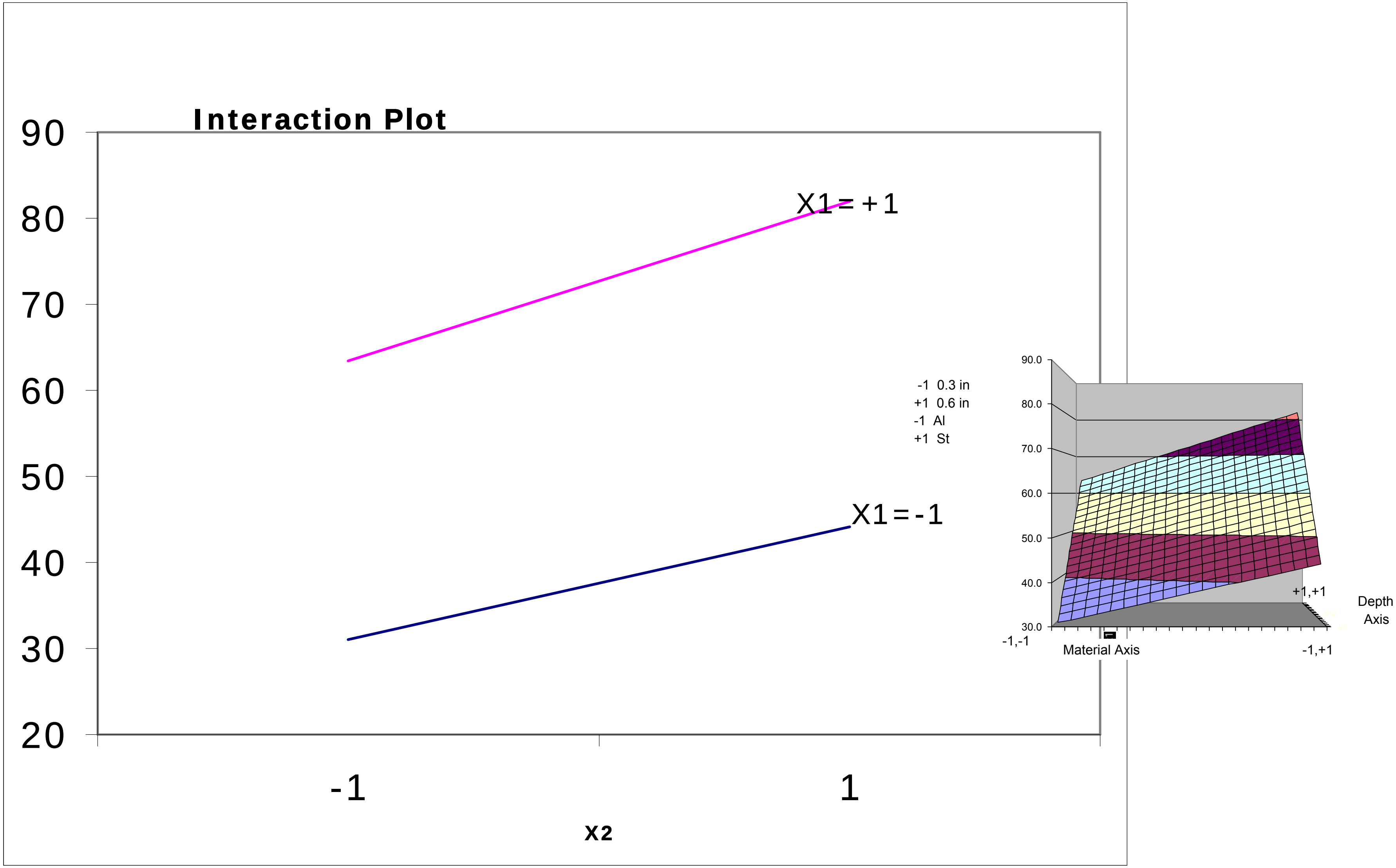
- Effect of Depth with Aluminum Only



Single Variable Plot: Material Effect



Interaction Effect?



ANOVA Test on Effects

Test	x1	x2	yi1	yi2	yi3	yi4	yi5	yi6	yi7	yi8	yi9	yi10
1	-1	-1	31.45	32.00	31.15	31.45	31.15	31.15	31.15	30.15	30.20	30.30
2	-1	1	45.30	45.10	45.00	42.15	44.00	45.35	44.55	43.30	44.30	42.15
3	1	-1	63.15	62.00	64.50	62.55	61.30	63.45	64.40	64.10	64.45	64.35
4	1	1	81.45	80.15	82.20	83.00	83.05	82.20	82.25	81.45	82.15	82.00

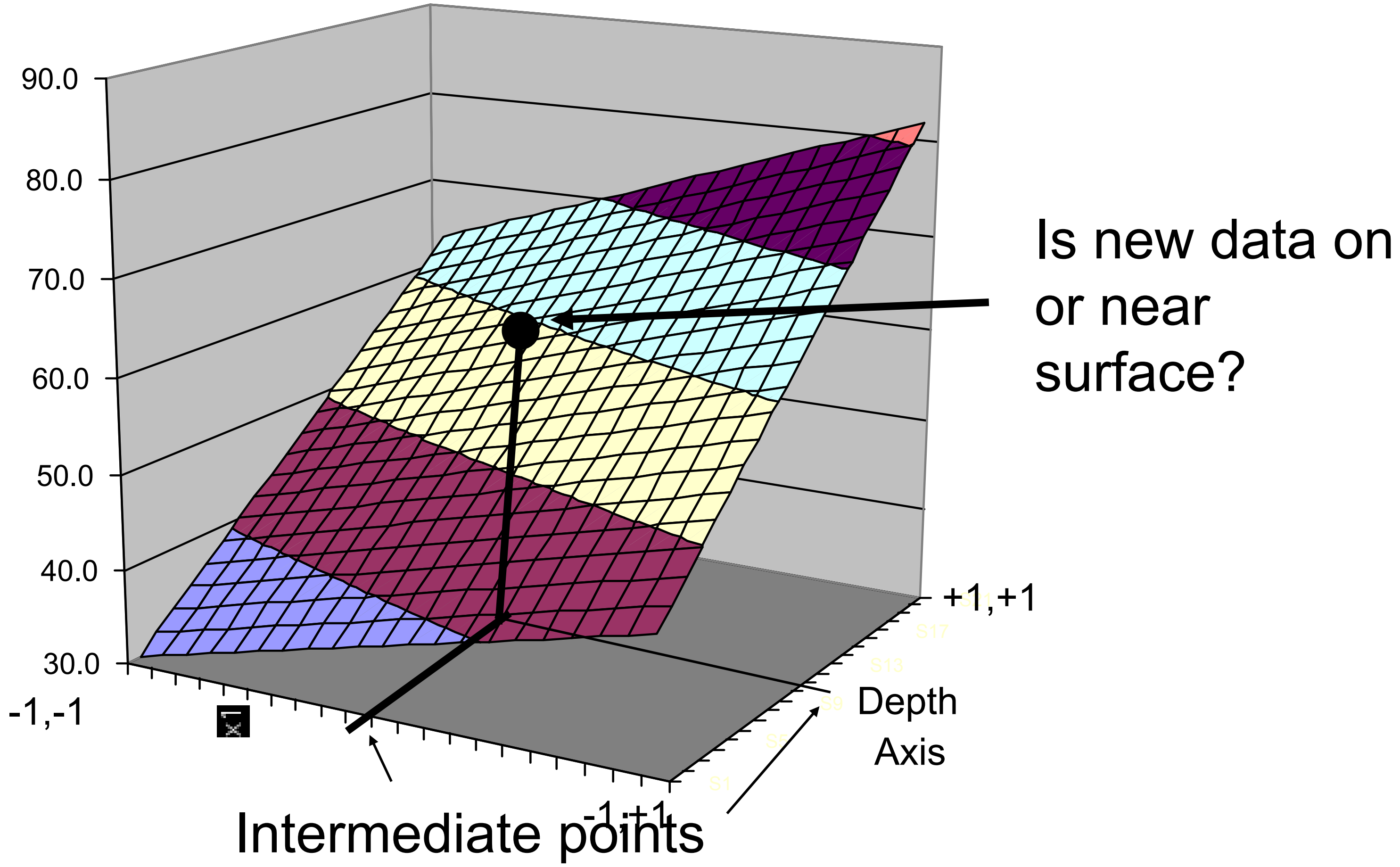
$$y = 55.1 + 17.6x_1 + 7.9x_2 + 1.4x_1x_2 + \varepsilon$$

mean	$nm\,\beta_0^{\,2}$	1	$\frac{SS(\beta_0)}{1}$	$\frac{MS(\beta_0)}{MS(\varepsilon)}$
x_1	$nm\,\beta_1^{\,2}$	1	$\frac{SS(\beta_1)}{1}$	$\frac{MS(\beta_1)}{MS(\varepsilon)}$
x_2	$nm\,\beta_2^{\,2}$	1	$\frac{SS(\beta_2)}{1}$	$\frac{MS(\beta_2)}{MS(\varepsilon)}$
x_{12}	$nm\,\beta_{12}^{\,2}$	1	$\frac{SS(\beta_{12})}{1}$	$\frac{MS(\beta_{12})}{MS(\varepsilon)}$
ε	$\sum_{i=1}^{m}\sum_{j=1}^{n}\varepsilon_{ij}$	$mn-4$	$\frac{SS(\varepsilon)}{(mn-4)}$	
total	$\sum_{i=1}^{m}\sum_{j=1}^{n}y_{ij}$	mn		

ANOVA on Effects			n=10	m=4	nm=40
	SS	DOF	MS	F	F (0.05)
mean	121606	1	1E+05	1E+05	4.1
X1	12348	1	1E+04	1E+04	4.1
X2	2507	1	2507.5	2593.8	4.1
X1X2	75	1	74.529	77.096	4.1
Error	35	36	0.9667		
Total	136571	40			

Is Model Form Adequate?

- How to Test?
  - Consider additional experimental (center) points



Questions and Hypotheses

- Lack of Fit Test: Is the Model Form Correct?
  - H_o: variance of lack of fit = pure (replicate) variance
  - H_1: variance of lack of fit ≠ pure (replicate) variance

- If H_0 the observed deviation from model prediction (e.g. at center point) could be explained by pure (replicate) error
  - Not enough evidence to attribute to model structure error

Testing for Quadratic Error

- Recall our Linear Model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + h.o.t. + \varepsilon$$

- Add a h.o.t.: $\beta_{11} x_1^{\ 2} + \beta_{22} x_2^{\ 2}$
- Check for deviation at center point
  - $x_1 = 0; x_2 = 0$
- What is our hypothesis
  - H_o: ?
  - H_1: ?

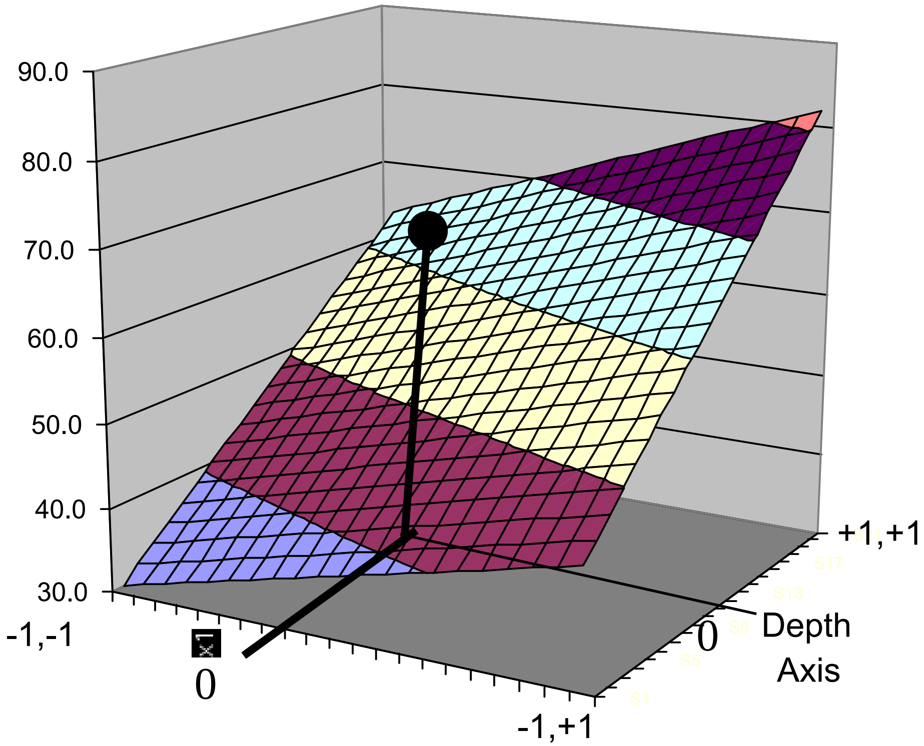
$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^{\ 2} + \beta_{22} x_2^{\ 2} + \varepsilon$$

Consider a Simple Test

Full Factorial linear with interactions, no replicates (n=1)

- Cannot test significance
- Cannot test for model fit

Add Central Point (A=0, B=0) with *n* replicates:



Use of Central Data

- Determine Deviation from Linear Prediction
  – Quadratic Term, or Central Error Term
- Determine MS of that Error
  – SS/dof
- Compare to Replication Error

Definitions

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x^2 + \beta_2 x^2$$

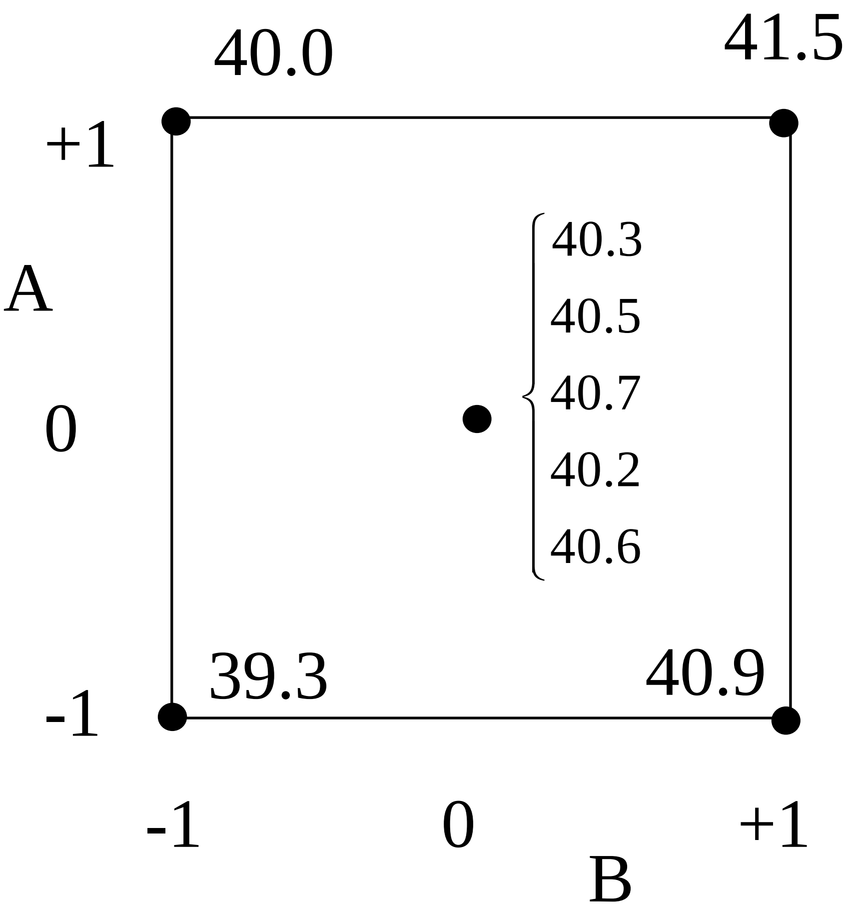
$$\bar{y}_F = \text{grand mean of all factorial runs}$$

$$\bar{y}_C = \text{grand mean of all center point runs}$$

$$SS_{Quadratic} = \frac{n_F n_C (\bar{y}_F - \bar{y}_C)^2}{n_F + n_C}$$

$$MS_{Quadratic} = \frac{SS_{Quadratic}}{n_c}$$

Example: 2^2 Without Replicates; Replicated Intermediate Points



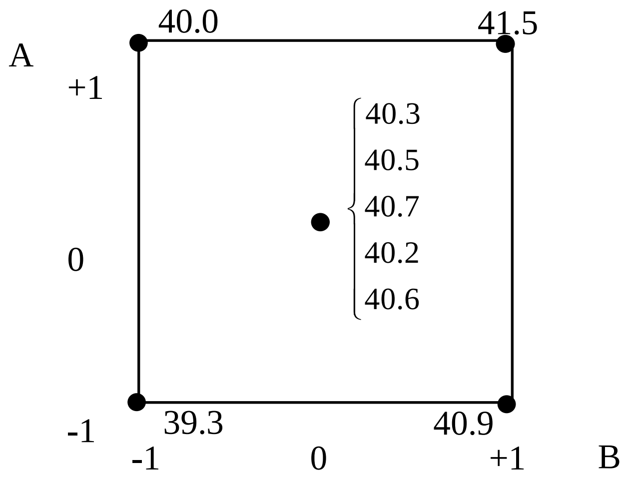
		I	A	B	AB
(1)	39.3	1	-1	-1	1
a	40.9	1	1	-1	-1
b	40	1	-1	1	-1
ab	41.5	1	1	1	1
Contrasts		161.7	3.1	1.3	-0.1
Effect		80.85	1.55	0.65	-0.05
Model Coefficients		40.43	0.775	0.325	-0.025

Just using corner points:

$$y = 40.43 + 0.775_1 x_1 + 0.325 x_2 - 0.025 x_1 x_2$$

Use of Central Data

(1)	40.3
a	40.5
b	40.7
ab	40.2
quad	40.6
Average	40.46
SS	0.172
Variance	0.04



ANOVA

Source	SS	DOF	MS	F	F(0.05)
A	2.4025	1	2.403	55.87	7.7
B	0.4225	1	0.423	9.83	7.7
AB	0.0025	1	0.003	0.06	7.7
Quad	0.002722	3	0.001	0.02	6.6
Error	0.172	4	0.043		
Total	3.002222	8			

Outline

- Full Factorial Models
  - Contrasts
  - Extension to 2^k
  - Model Term Significance: ANOVA
  - Checking Adequacy of Model Form
    - Tests for higher order fits (curvature)
- Experimental Design
  - Blocks and Confounding
  - Single Replicate Designs
  - Fractional Factorial Designs

Experimental Design Issues

- Nuisance Factors
  - Affect the output, but don't want the effect
  - May not be able to run whole experiment holding that factor constant
- If Known but Uncontrollable
  - Randomization: treat as random "noise" factor
- If Known and Controllable
  - Separate data into block where nuisance is constant
  - E.g., if replicating, run each replicate of full design with same block factor

Replicated Block Design

Hardness Test

Tip	Test Coupon 1	2	3	4
1	9.3	9.4	9.6	10.0
2	9.4	9.3	9.8	9.9
3	9.2	9.4	9.5	9.7
4	9.7	9.6	10.0	10.2

Each Block is like a Factor

Randomize Order Within Blocks

$$y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \begin{cases} i = 1, 2, 3 \ldots a \\ j = 1, 2, 3, b \end{cases}$$

Nonreplicated Block Design

- Suppose 2^2 design
  - 2 factors, 2 levels each = 4 runs
  - But we have to arrange to do 2 runs (block 1), then another 2 runs (block 2)
    - Expect an unknown offset Δ between block 1 & 2
- Question: How arrange runs?

	X_1	X_2	
(1)	–	–	Block 1
a	+	–	
b	–	+	Block 1
ab	+	+	

Can't distinguish X_2 effect from Δ!

Nonreplicated Block Design

Better approach:

Block 1 = (1) and *ab*
Block 2 = *a* and *b*

$$Contrast_A = [a + ab - b - (1)]$$

$$Contrast_B = [b + ab - a - (1)]$$

Δ's within each block

$$Contrast_{AB} = [ab + (1) - a - b]$$

Δ's across each block

Blocking and Confounding

$$Contrast_A = [a + ab - b - (1)] = (ab - (1)) + (b - a)$$
$$Contrast_B = [b + ab - a - (1)] = (ab - (1)) + (a - b)$$
$$Contrast_{AB} = [ab + (1) - a - b] = (ab + (1)) - (a + b)$$

Now assume the block effect is that block 2 has an offset of δ from what it would be if done in block 1

$$a = a + \delta$$
$$b = b + \delta$$

This gives

$$Contrast_A = (ab - (1)) + (b + \delta - a - \delta)$$ δ's cancel

$$Contrast_B = (ab - (1)) + (a + \delta - b - \delta)$$ δ's cancel

$$Contrast_{AB} = (ab + (1)) - (a + \delta + b + \delta)$$ δ's double!

$2\,\delta$

Fractional Factorial Experiments

- What if we do less than full factorial 2^k?
- From regression model for 3 inputs:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_3 x_3$$
$$+ \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123z} x_1 x_2 x_3 + \varepsilon$$

- We will not be able to find all 8 coefficients

2^{3-1} Experiment

- Consider doing 4 experiments instead of 8; e.g.:

	x_1	x_2	$x_1 x_2$
1	-1	-1	$+1$
2	$+1$	-1	-1
3	-1	$+1$	-1
4	$+1$	$+1$	$+1$

- This is a 2^2 array
- Could also be for 3 inputs if we define $x_3 = x_1 x_2$

2^{3-1} Experiment

	x_1	x_2	x_3
1	-1	-1	$+1$
2	$+1$	-1	-1
3	-1	$+1$	-1
4	$+1$	$+1$	$+1$

But now we can only define 4 coefficients in the model: e.g.:

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$

i.e. no interaction terms

2^{3-1} Experiment

Or we could choose other terms:

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{13} x_1 x_3$$

or:

$$\hat{y} = \beta_0 + \beta_1 x_1 + \beta_{12} x_1 x_2 + \beta_3 x_3$$

or:

…

Confounding / Aliasing

- We actually have the following:

$$\hat{y} = \beta_0 + \beta'_1 z_1 + \beta'_2 z_2 + \beta'_3 z_3$$

- where the *z* variable represent sums of the various input terms, e.g.

$$z_1 = x_1 x_2 + x_3$$

$$z_2 = x_1 + x_2 x_3 \ldots$$

- where the specific choice of the experimental array determines what these sums are

Confounding / Aliasing

2^3 Array: (Our **X** matrix)

Test	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

Confounding / Aliasing

Consider upper half:

Test	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

Look at columns for C - no change at all! or C = -I

Also AC = -A and BC = -B, and ABC = -AB

Confounding / Aliasing

Test	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

$Contrast_A$=[-(1)+a-b+ab]

AC is an alias of A

$Contrast_{AC}$=[(1)-a+b-ab]

Note that alias of A =A*(-C)

Defining Relation I = -C

Choice of Design?

- Aliases
  - Must have one of the pair assumed negligible ("sparsity of effects")
- Balance/Orthogonality
  - Sufficient excitation of inputs

Balance and Orthogonality

Test	I	A	B	AB	C	AC	BC	ABC
(1)	1	-1	-1	1	-1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
ab	1	1	1	1	-1	-1	-1	-1
c	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	-1	1	1	-1	-1
bc	1	-1	1	-1	1	-1	1	-1
abc	1	1	1	1	1	1	1	1

Note: All columns have equal number of + and - signs (Balance)
Sum of product of any two columns = 0 (Orthogonality)
-All combinations occur the same number of times

Balance/Orthogonality in 2^{3-1}

Test	I	A	B	C	AB	AC	BC	ABC
1	1	-1	-1	-1	1	1	1	-1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
c	1	1	1	-1	1	-1	-1	-1
ab	1	-1	-1	1	1	-1	-1	1
ac	1	1	-1	1	-1	1	-1	-1
bc	1	-1	1	1	-1	-1	1	-1
abc	1	1	1	1	1	1	1	1

A, B and C are balanced but B and C are not orthogonal

Design Resolution

Test	I	A	B	C	AB	AC	BC	ABC
1	1	-1	-1	1	1	-1	-1	1
a	1	1	-1	-1	-1	-1	1	1
b	1	-1	1	-1	-1	1	-1	1
c	1	1	1	1	1	1	1	1

With this array:

-balance for A, B, C

-all but A B C are orthogonal

-defining relation I=ABC

e.g. aliases of A:
A*ABC=A*I
A*A = I
BC aliased with A

Main effects aliased with interactions only →

Aliases:

A BC

B AC

C AB

I ABC

Design Resolution

- Resolution III
  - No Main aliases
  - Main - Interaction Aliases
- Resolution IV
  - No Alias between main effects and 2 factor effects, but others exist
- Resolution V
  - No Main and no 2 Factor Aliases ….

Smaller Fraction 2^{k-p}

- p = 1 1/2 fraction
- p= 2 1/4 fraction
- p $1/2^p$

2^{4-2}

	A	B	C	D
1	-1	-1	-1	-1
2	1	-1	-1	-1
3	-1	1	-1	-1
4	1	1	-1	-1
5	-1	-1	1	-1
6	1	-1	1	-1
7	-1	1	1	-1
8	1	1	-1	-1
9	-1	-1	-1	1
10	1	-1	-1	1
11	-1	1	-1	1
12	1	1	-1	1
13	-1	-1	1	1
14	1	-1	1	1
15	-1	1	1	1
16	1	1	1	1

Four Main Effects
Four tests?

Suppose we want to alias A with BCD and ABC

What are the defining relations?

Summary

- Full Factorial Models
  - Contrasts
  - Extension to 2^k
  - Model Term Significance: ANOVA
  - Checking Adequacy of Model Form
    - Tests for higher order fits (curvature)
- Experimental Design
  - Blocks and Confounding
  - Single Replicate Designs
  - Fractional Factorial Designs