Exercises on solving $Ax = b$ and row reduced form $R$

Problem 8.1:  (3.4 #13.(a,b,d) Introduction to Linear Algebra: Strang) Explain why these are all false:

a) The complete solution is any linear combination of $x_p$ and $x_n$.

b) The system $Ax = b$ has at most one particular solution.

c) If $A$ is invertible there is no solution $x_n$ in the nullspace.

Solution:

a) The coefficient of $x_p$ must be one.

b) If $x_n \in N(A)$ is in the nullspace of $A$ and $x_p$ is one particular solution, then $x_p + x_n$ is also a particular solution.

c) There’s always $x_n = 0$.

Problem 8.2:  (3.4 #28.) Let

$$U = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}.$$

Use Gauss-Jordan elimination to reduce the matrices $[U \ 0]$ and $[U \ \mathbf{c}]$ to $[R \ 0]$ and $[R \ \mathbf{d}]$. Solve $Rx = 0$ and $Rx = \mathbf{d}$.

Check your work by plugging your values into the equations $Ux = 0$ and $Ux = \mathbf{c}$.

Solution:  First we transform $[U \ 0]$ into $[R \ 0]$:

$$[U \ 0] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [R \ 0].$$

We now solve $Rx = 0$ via back substitution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + 2x_2 = 0 \\ x_3 = 0 \end{bmatrix} \longrightarrow x = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},$$
where we used the free variable $x_2 = -1$. ($cx$ is a solution for all $c$.)

We check that this is a correct solution by plugging it into $UX = 0$:

$$
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
2 \\
-1 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\checkmark
$$

Next, we transform $[U \ c]$ into $[R \ d]$:

$$
[U \ c] = 
\begin{bmatrix}
1 & 2 & 3 & 5 \\
0 & 0 & 4 & 8
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
1 & 2 & 3 & 5 \\
0 & 0 & 1 & 2
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
1 & 2 & 0 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix} = [R \ d].
$$

We now solve $Rx = d$ via back substitution:

$$
\begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
-1 \\
2
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
x_1 + 2x_2 = -1 \\
x_3 = 2
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
-3 \\
1 \\
2
\end{bmatrix},
$$

where we used the free variable $x_2 = 1$.

Finally, we check that this is the correct solution by plugging it into the equation $UX = c$:

$$
\begin{bmatrix}
1 & 2 & 3 \\
0 & 0 & 4
\end{bmatrix}
\begin{bmatrix}
-3 \\
1 \\
2
\end{bmatrix}
= 
\begin{bmatrix}
5 \\
8
\end{bmatrix}
\checkmark
$$

**Problem 8.3:** (3.4 #36.) Suppose $Ax = b$ and $Cx = b$ have the same (complete) solutions for every $b$. Is it true that $A = C$?

**Solution:** Yes. In order to check that $A = C$ as matrices, it is enough to check that $Ay = Cy$ for all vectors $y$ of the correct size (or just for the standard basis vectors, since multiplication by them “picks out the columns”). So let $y$ be any vector of the correct size, and set $b = Ay$. Then $y$ is certainly a solution to $Ax = b$, and so by our hypothesis must also be a solution to $Cx = b$; in other words, $Cy = b = Ay$.  

2