Transposes, permutations, spaces \( \mathbb{R}^n \)

In this lecture we introduce vector spaces and their subspaces.

**Permutations**

Multiplication by a permutation matrix \( P \) swaps the rows of a matrix; when applying the method of elimination we use permutation matrices to move zeros out of pivot positions. Our factorization \( A = LU \) then becomes \( PA = LU \), where \( P \) is a permutation matrix which reorders any number of rows of \( A \). Recall that \( P^{-1} = P^\top \), i.e. that \( P^\top P = I \).

**Transposes**

When we take the transpose of a matrix, its rows become columns and its columns become rows. If we denote the entry in row \( i \) column \( j \) of matrix \( A \) by \( A_{ij} \), then we can describe \( A^\top \) by:

\[
\begin{bmatrix}
1 & 3 \\
2 & 3 \\
4 & 1
\end{bmatrix}^\top =
\begin{bmatrix}
1 & 2 & 3 & 4 \\
3 & 1
\end{bmatrix}.
\]

A matrix \( A \) is symmetric if \( A^\top = A \). Given any matrix \( R \) (not necessarily square) the product \( R^\top R \) is always symmetric, because \( (R^\top R)^\top = R^\top (R^\top)^\top = R^\top R \). (Note that \( (R^\top)^\top = R \).)

**Vector spaces**

We can add vectors and multiply them by numbers, which means we can discuss linear combinations of vectors. These combinations follow the rules of a vector space.

One such vector space is \( \mathbb{R}^2 \), the set of all vectors with exactly two real number components. We depict the vector \( \begin{bmatrix} a \\ b \end{bmatrix} \) by drawing an arrow from the origin to the point \( (a, b) \) which is \( a \) units to the right of the origin and \( b \) units above it, and we call \( \mathbb{R}^2 \) the “\( x-y \) plane”.

Another example of a space is \( \mathbb{R}^n \), the set of (column) vectors with \( n \) real number components.

**Closure**

The collection of vectors with exactly two positive real valued components is not a vector space. The sum of any two vectors in that collection is again in the collection, but multiplying any vector by, say, \( -5 \), gives a vector that’s not
in the collection. We say that this collection of positive vectors is closed under addition but not under multiplication.

If a collection of vectors is closed under linear combinations (i.e. under addition and multiplication by any real numbers), and if multiplication and addition behave in a reasonable way, then we call that collection a vector space.

**Subspaces**

A vector space that is contained inside of another vector space is called a subspace of that space. For example, take any non-zero vector \( v \) in \( \mathbb{R}^2 \). Then the set of all vectors \( cv \), where \( c \) is a real number, forms a subspace of \( \mathbb{R}^2 \). This collection of vectors describes a line through \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) in \( \mathbb{R}^2 \) and is closed under addition.

A line in \( \mathbb{R}^2 \) that does not pass through the origin is not a subspace of \( \mathbb{R}^2 \). Multiplying any vector on that line by 0 gives the zero vector, which does not lie on the line. Every subspace must contain the zero vector because vector spaces are closed under multiplication.

The subspaces of \( \mathbb{R}^2 \) are:

1. all of \( \mathbb{R}^2 \),
2. any line through \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and
3. the zero vector alone (Z).

The subspaces of \( \mathbb{R}^3 \) are:

1. all of \( \mathbb{R}^3 \),
2. any plane through the origin,
3. any line through the origin, and
4. the zero vector alone (Z).

**Column space**

Given a matrix \( A \) with columns in \( \mathbb{R}^3 \), these columns and all their linear combinations form a subspace of \( \mathbb{R}^3 \). This is the column space \( C(A) \). If \( A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \),

the column space of \( A \) is the plane through the origin in \( \mathbb{R}^3 \) containing \( \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \)

and \( \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \).

Our next task will be to understand the equation \( Ax = b \) in terms of subspaces and the column space of \( A \).
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