18.06SC Final Exam

1 (4+7=11 pts.) Suppose A is 3 by 4, and $Ax = 0$ has exactly 2 special solutions:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Remembering that A is 3 by 4, find its row reduced echelon form R.

(b) Find the dimensions of all four fundamental subspaces $C(A)$, $N(A)$, $C(A^{\mathrm{T}})$, $N(A^{\mathrm{T}})$.

You have enough information to find bases for one or more of these subspaces—find those bases.

2 (6+3+2=11 pts.) (a) Find the inverse of a 3 by 3 upper triangular matrix U, with **nonzero** entries a, b, c, d, e, f. You could use cofactors and the formula for the inverse. Or possibly Gauss-Jordan elimination.

Find the inverse of $U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

(b) Suppose the columns of U are eigenvectors of a matrix A. Show that A is also upper triangular.

(c) Explain why this U **cannot** be the same matrix as the first factor in the Singular Value Decomposition $A = U\Sigma V^{\mathrm{T}}$.

3 (3+3+5=11 pts.) (a) A and B are any matrices with the same number of rows. What can you say (*and explain why it is true*) about the comparison of

rank of A rank of the block matrix $\begin{bmatrix} A & B \end{bmatrix}$

(b) Suppose $B = A^2$. How do those ranks compare? Explain your reasoning.

(c) If A is m by n of rank r, what are the dimensions of these nullspaces?

Nullspace of A Nullspace of $\begin{bmatrix} A & A \end{bmatrix}$

4 (3+4+5=12 pts.) Suppose A is a 5 by 3 matrix and Ax is never zero (except when x is the zero vector).

(a) What can you say about the columns of A?

(b) Show that $A^{\mathrm{T}}Ax$ is also never zero (except when $x = 0$) by explaining this key step:

If $A^{\mathrm{T}}Ax = 0$ then obviously $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$ and then (WHY ?) $Ax = 0$.

(c) We now know that $A^{\mathrm{T}}A$ is invertible. Explain why $B = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is a one-sided inverse of A (which side of A?). B is NOT a 2-sided inverse of A (*explain why not*).

5 (5+5=10 pts.) If A is 3 by 3 symmetric positive definite, then $Aq_i = \lambda_i q_i$ with positive eigenvalues and orthonormal eigenvectors q_i.

Suppose $x = c_1 q_1 + c_2 q_2 + c_3 q_3$.

(a) Compute $x^{\mathrm{T}} x$ and also $x^{\mathrm{T}} A x$ in terms of the c's and λ's.

(b) Looking at the ratio of $x^{\mathrm{T}} A x$ in part (a) divided by $x^{\mathrm{T}} x$ in part (a), what c's would make that ratio as large as possible? You can assume $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. Conclusion: the ratio $x^{\mathrm{T}} A x / x^{\mathrm{T}} x$ is a maximum when x is ______.

6 (4+4+4=12 pts.) (a) Find a linear combination w of the linearly independent vectors v and u that is perpendicular to u.

(b) For the 2-column matrix $A = \begin{bmatrix} u & v \end{bmatrix}$, find Q (orthonormal columns) and R (2 by 2 upper triangular) so that $A = QR$.

(c) In terms of Q only, using $A = QR$, find the projection matrix P onto the plane spanned by u and v.

7 (4+3+4=11 pts.) (a) Find the eigenvalues of

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

(b) Those are both permutation matrices. What are their inverses C^{-1} and $(C^2)^{-1}$?

(c) Find the determinants of C and $C + I$ and $C + 2I$.

8 (4+3+4=11 pts.) Suppose a rectangular matrix A has independent columns.

(a) How do you find the best least squares solution $\widehat{x}$ to $Ax = b$? By taking those steps, give me a formula (letters not numbers) for $\widehat{x}$ and also for $p = A\widehat{x}$.

(b) The projection p is in which fundamental subspace associated with A? The error vector $e = b - p$ is in which fundamental subspace?

(c) Find by any method the projection matrix P onto the column space of A:

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}.$$

9 (3+4+4=11 pts.) This question is about the matrices with 3's on the main diagonal, 2's on the diagonal above, 1's on the diagonal below.

$$A_1 = \begin{bmatrix} 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad A_n = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix}$$

(a) What are the determinants of A_2 and A_3?

(b) The determinant of A_n is D_n. Use cofactors of row 1 and column 1 to find the numbers a and b in the recursive formula for D_n:

$$(*) \qquad D_n = a\,D_{n-1} + b\,D_{n-2}\,.$$

(c) This equation $(*)$ is the same as

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}.$$

From the eigenvalues of that matrix, how fast do the determinants D_n grow? (If you didn't find a and b, say how you would answer part (c) for any a and b) For 1 point, find D_5.