Origami terminology:

**Piece of paper** = 2D polygon (most often) with distinguished top/bottom sides

**Crease** = line segment or curve on paper

**Crease pattern** = collection of creases = planar graph drawn on paper

**Folded state** = finished origami

- unfolding → crease pattern

**Flat folding** = folded state lying in a plane

- call its crease pattern flat foldable  
  (must use all creases)

Mountain crease = bottom sides touch

Valley crease = top sides touch

Mountain-valley assignment = which creases in crease pattern are mountain/valley origami notation: --/----

Mountain-valley pattern = crease pattern + mountain-valley assignment

Example: crumple up a piece of paper
Simple fold: fold along a single line, by $\pm 180^\circ$ (mountain/valley)
- choice of how many layers to fold

One-layer simple fold = just top or bottom
All-layers simple fold = all the way through

Today is mostly about simple folds

Strip: long narrow rectangle

Example of nonsimple strip folding:
tie a knot
Every connected union of polygons in 3D, each with a specified visible color (on each side), can be folded from a sufficiently large piece of bicolor paper of any shape (e.g., square).

Proof:
- fold paper down to long narrow strip (!)
  - triangulate the polygons
  - choose a path visiting each triangle at least once
  - cover each triangle along the path by zig-zag parallel to next edge, starting at opposite corner:

  ![Diagram of zig-zag folding process]

  - choose parity of zig-zag to arrive at correct corner for next triangle

- turn gadget implements zig-zags & vertex turns:

  ![Diagrams of zig-zag turns]

  1. perpendicular mountain
  2. fold bottom layer
  3. hide excess (many folds)
Proof of folding any shape: (cont’d)

- hide excess paper underneath each triangle:
  (more generally, can hide under any convex polygon)

- if paper is unicolor (or don’t care)
  can use valley folds ⇒ simple folds

- if mountain folds, might collide with other Δs ⇒ not really simple folds
  (but still works as origami fold)

- color-reversal gadget along transition between triangles of opposite colors:

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① ← ② ③ ④
fold bottom layer
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Pseudo-efficiency: if allowed to start with any rectangle of paper, then can achieve $\text{area(paper)} = \text{area(surface)} + \varepsilon$ for any $\varepsilon > 0$.

Proof: construct Hamiltonian refinement of triangulation:
- cut each $\triangle$ into
- walk around spanning tree of original dual:
  - now visit each triangle exactly once
  - wastage from turns $\rightarrow 0$ with strip width.

Alternate proof from class: (should work)
- visit $\Delta$s in any order, but when traversing from one to next, go directly (without covering (much of) intervening $\Delta$s)
- wastage is $\approx$ strip width $\cdot \leq$ diameter($\Delta$)
  $\rightarrow 0$ as strip width $\rightarrow 0$

OPEN: pseudopolynomial upper bound? lower bound?
Seam placement: can place seams (visible creases/paper boundary) as desired, provided regions between seams are convex.
- Idea: vary strip width, use hide gadget.

OPEN: what seam placements are possible?
1D flat folding: [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena 2004]

Piece of paper = line segment
Crease = point on paper
Flat folding lies on a line

All crease patterns are flat foldable:
zig-zag / accordion fold

Not all mountain-valley patterns:

Two folding operations: (both simple)
1. end fold if end length $\le$ neighbor
2. crimp two consecutive creases if length between $\le$ both neighbors & one mountain, one valley

Characterization:
mountain-valley pattern is flat foldable $\iff$ there's a sequence of crimps & end folds

Tool: Mingling: for any maximal sequence of M's or V's, adjacent V or M or end on at least one side is nearer than adjacent M or V:

OPEN: proof without mingling?
1D flat folding (cont'd)

Proof of characterization:
- flat foldable $\Rightarrow$ mingling:
  - sequence of M's or V's form a spiral (or double spiral)
  - at least one end must be short
- mingling $\Rightarrow$ end fold or crimp possible
  - for each maximal sequence of M's or V's write ( if "left mingling"
    [ otherwise
  & write ) if "right mingling"
    [ otherwise
  $\Rightarrow$ ( [ ] or [ ] ) or ( )
  - ) ( $\Rightarrow$ crimp
  - leading ( / trailing ) $\Rightarrow$ end fold
  - if neither: [ ] [ ] ... [ ] [ ] [ ]... [ ]
    $\Rightarrow$ crash
- crimp/end fold preserves flat foldability
  - take flat folding
  - move some layers out of crimp
  $\Rightarrow$ could start with crimp
- induct $\Rightarrow$ sequence of crimps & end folds
  $\Rightarrow$ flat foldable again.
2D map folding: [Arkin et al. 2004]
- rectangular paper with axis-parallel creases
- again every crease pattern is flat foldable:
  zig-zag in x then y

OPEN: characterize flat-foldable mountain-valley patterns — even 2×n! [Edmonds 1997]

Simple folds are not as powerful in 2D:
  (in contrast to 1D, where we can simulate crimp/end folds)

Characterization of simple foldability of maps:
- if simply foldable, must be a uniform horizontal/vertical line
  \( \Rightarrow \) all M or all V
- crossing vert./horiz. lines must switch M\( \Leftrightarrow \)V here:
  - local 2×2 patterns: & rotations

- all uniform horizontal lines must be folded before any vertical lines become uniform, etc.
- sequence of 1D problems on current uniform lines
- linear-time algorithm — \( O(mn) \) for \( m \times n \) map —
  by maintaining uniformity for each line,
  crimpability & end foldability on lengths