Universal hinge patterns: (for origami transformers) [Benbernou, Demaine, Demaine, Ovadia 2010]
- Suppose crease pattern required to be subset of fixed "hinge pattern"
  (e.g. Origamizer uses completely different creases for every model)
- n x n box-pleat pattern can make any polycube of $O(n)$ cubes, seamless:
  - Cube gadget turns $O(1)$ rows & columns into a cube sticking out of sheet ~
    even if bumps elsewhere (not in eaten rows/cols.)
- To make a tree of cubes: (= any polycube)
  - Make a leaf
  - Conceptually remove it
  - Repeat
- Actually need to reserve space ahead of time for all the cube gadgets
- $\Theta(n)$ cubes is optimal in worst case: $1 \times 1 \times n$ needs diameter $\Omega(n)$
- but sometimes can do better:

**Maze folding:** [Demaine, Demaine, Ku 2010]
any $n \times n$ orthogonal maze extruded from square can be folded from $\Theta(n) \times \Theta(n)$ square
- constant scale factor! (3 for unit extrusion)
- gadget for each possible vertex:

```
    .   .   .   .   .
    0   1   2   2   3   4
```
degree 0 1 2 2 3 4
- designed to have compatible interfaces:
- ridge for maze edges
- flat "double pleat" for nonedges
- cut & paste

try it out: http://erikdemaine.org/maze/
Origami design is hard ~ how to formalize?

**NP-hard** ≈ “computationally intractable”
- if a problem is NP-hard, then there’s no efficient algorithm to solve it unless \( P=NP \)
  (famous unsolved problem, worth $1M+)
- \( P \neq NP \) ≈ “computers can’t simulate lucky guessing, say heads vs. tails, without trying both options”
  \( \Leftrightarrow \) almost everyone believes it

Examples of NP-hard problems:
- **Partition**: given \( n \) integers, can you split them into two halves of equal sum?
  (e.g. equalizing teams for a game)
  - actually only hard for exp. large integers: “weakly NP-hard”
- **SAT**: given Boolean formula \((x \text{ AND NOT } y) \text{ OR } z\)
  can you set the variables \( x, y, z \) true/false so that formula is true?

Approach: show e.g. Partition is easier/a special case of your problem: any Partition problem can be converted into a problem of your type
\( \Rightarrow \) your problem is NP-hard too
Simple example: (from Problem Session 1)
given single-vertex hinge pattern, is some subset of (>Ø) creases flat foldable? (posed by student after class)
is NP-hard:
- given Partition problem, scale integers uniformly so that their sum = 360°
  → angles of single-vertex crease pattern
- looking at angular travel (Kawasaki-Justin), at each hinge can crease → change direction or not → same direction
  ⇒ can choose +θᵢ or -θᵢ for each i
- must have Σᵢ ± θᵢ = Ø
  i.e. Σᵢ + θᵢ’s = Σᵢ - θᵢ’s
- YES to Partition ⇔ YES to flat-foldable CP
Simple folds: can given crease pattern be folded flat by sequence of simple folds?
- saw how to solve for 1D patterns & 2D orthogonal maps:

NP-hard if we add 45° diagonal creases or allow orthogonal paper
[Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena 2000]
- reduction from Partition (weakly NP-complete)

\[ \frac{1}{2} \sum_{i=1}^{n} a_i \geq L \]
- fold some of horiz. creases
- fold a vertical crease
- if hit wall: stuck
- else can fold other vertical & remaining horiz. creases
Global flat foldability: [Bern & Hayes 1996]

1. deciding flat foldability of given crease pattern is strongly NP-hard
2. constructing valid layer ordering for given flat-foldable mountain-valley pattern is strongly NP-hard

Proof: (1) reduce from all-positive not-all-equal 3-satisfiability: given triples \( (x_i, x_j, x_k) \), is there a Boolean assignment to \( x_1, x_2, \ldots, x_n \) such that no triple is all-true or all-false? (strongly NP-hard, like SAT)

\[ \text{Wire} = \text{"pleat" = two close parallel creases} \]
\[ \text{false } \Leftrightarrow \text{left mountain} \quad \Rightarrow \quad \text{false} \quad \Rightarrow \quad \text{true} \]

\[ \text{NAE clause = triangular \"overtwist\"} \]
\[ \text{can't all fold same way} \]
\[ \text{(twist is borderline)} \]

Reflector splits wire \( x \) into two copies, one negated
\[ \Rightarrow \text{split gadget } \uparrow \uparrow \]
\[ \& \text{turn gadget } \hspace{1cm} \Rightarrow \text{(with noise)} \]
\[ \Rightarrow \text{can connect variable wires to desired clauses} \]

Also need crossover gadgets. \( \square \)
**Disk packing:** [Demaine, Fekete, Lang 2010]

can you place n given disks nonoverlapping with centers in given square?  

= can you make uniaxial base from given square?  

is NP-hard

- reduction from 3-Partition:  
given n integers, can you split them into \( \frac{n}{3} \) triples of equal sum?  
  - strongly NP-hard \( \sim \) integers = \( O(n) \), not exp.

- lots of disks to force identical pockets & make all other pockets too small

- within one pocket:

\[
\begin{array}{c}
\text{a}_i \text{ oversized} \\
\text{undersized by desired triple sum} \\
\text{a}_j \text{ oversized} \\
\text{a}_k \text{ oversized} \\
\Rightarrow \text{just fits if 3-partition} \\
\end{array}
\]