Fold & one cut:
1. fold flat
2. make one complete straight cut
3. unfold

- what shapes/patterns of cuts are possible?

“Wakoku Chiye Kurabe”

History:
- Betsy Ross [1873 story] - ★ in American flag
- Gerald Lee [1955] - Paper Capers ~ simple folds
- Martin Gardner [1960] - Scientific American ~ OPEN

Universality: any set of line segments can be cut
- two methods:
  1. straight skeleton [Demaine, Demaine, Lubiw 1998]
     - works almost always: practical
  2. disk packing [Bern, Demaine, Eppstein, Hayes 1998]
    - always works; pseudopolynomial; impractical

Warm-ups:
1 line
- no folds

2 lines
- bisector

triangle
- angle bisection + perpendicular
**Straight skeleton:** [Aichholzer et al. 1995 & 1996]

= trajectory of the vertices of the desired cut pattern as we simultaneously shrink each region, keeping edges parallel to the originals & at uniform perpendicular distance

**Events during shrinking:**
1. edge shrinks to Ø length  \( \Rightarrow \) drop it
2. entire region collapses to Ø area  \( \Rightarrow \) add it all & stop shrinking it
3. face “splits”  \( \Rightarrow \) recurse in pieces

**Degree-1 vertex** like end of a rectangle

**Degree-Ø vertex** like a square

**Facts:**
- \( O(n) \) skeleton vertices, edges, regions
- one-to-one correspondence between cut edges and regions of the straight skeleton
- every skeleton edge is a subsegment of the (angular) bisector of the cut edges corresponding to the two incident skeleton regions  \( \Rightarrow \) align
Generic skeleton vertex has degree 3 ⇒ not flat foldable
⇒ need to add some creases...

**Perpendiculars:** [Demaine, Demaine, Lubiw 1998]
- add creases that meet desired cuts at right angles ⇒ preserve alignment
- from each skeleton vertex, try to enter each incident skeleton region with ray perpendicular to corresponding cut edge
- if ray hits another skeleton edge, reflect (⇒ remains perpendicular to corresponding cut edge)

**Typical behavior at skeleton vertex now:**
- skeleton edges bisect perpendiculars
⇒ Kawasaki condition holds

**Spiraling:**
⇒ infinite creases, but finite in finite paper

**Density:**
⇒ creases are dense

**Conjecture:** rare (prob. 0)
Mountain-valley assignment: (initial)
- skeleton edge mountain if bisects convex angle
  valley if bisects reflex angle
- cut edge valley
- perpendiculars alternate mountain/valley;
  start to be determined later

Side assignment: specify which cut regions are above or below the cut line
- skeleton edges as above in above regions;
  reversed in below regions
- cut edge valley between two above regions
  mountain between two below regions
  uncreased between one above & one below

- e.g. 2-regular (nested/disjoint polygons)
  ⇒ natural 2-coloring
  ⇒ all cuts uncreased ("scissor cuts")
- e.g. 4-regular checkerboard

PROJECT: implement crease pattern algorithm
  (ideally with degeneracy tool, MV assignment, folded state...)

Send me your cool fold & cut examples!
Corridor = region delimited by perpendiculars (like rivers)
- constant width, measured perpendicularly
- either **linear** = eventually reach infinity
  or **circular** = closed loop
  \[ \Rightarrow \text{harder to fold (theoretically & practically)} \]
- **CONJECTURE**: max. degree 2 \( \Rightarrow \) linear corridors only
  with probability 1

**Linear-corridor case**: (proof sketch)
- each corridor folds as an accordion
  - alternates mountain/valley
  - aligns cut edges
- corridors form a tree structure \( \simeq \) projection
  - edge = corridor, length = width
  - vertex = connected component of perpendiculars
- fold tree flat by depth-first search
  \( \Rightarrow \) origami folds flat (argue noncrossing)

**Circular-corridor case**:
- trouble: accordion has to wrap around at some edge — reversed; intersect?
- **CONJECTURE**: with probability 1, circular corridors are normal \( \Rightarrow \)
- if normal & side assignment is "all above"
  then can reverse a cut edge:
Disk-packing method: [Bern, Demaine, Eppstein, Hayes 1998-] & O'Rourke

1. Thicken desired cuts by 2\(\varepsilon\) by parallel offset by \(\pm \varepsilon\) (\(\varepsilon\) suff. small) (just like straight skeleton)

2. Find a (nonoverlapping) disk packing such that
   a. Every vertex of offset cuts & paper boundary is the center of a disk
   - Put small disk at each vertex
   b. Every edge of ... is a union of radii
   - Pack small disks along each edge
   c. Every gap between disks has 3 or 4 sides
   - Repeatedly subdivide gaps:
      [Eppstein 1997]

3. Dual \(\Rightarrow\) decomposition into triangles & quads.
4. Fold each triangle/quad into molecule aligning its boundary

5. Glue molecules together \(\Rightarrow\) align all edges
   - Argue no crossings \(\sim\) hard part
6. Sink-fold exterior molecules to height \(< \varepsilon\)
   for single polygon case
Disk-packing method: (cont’d)
- can generalize to arbitrary cut graphs
  (but not arbitrary side assignments)
  - joining & sinking gets messier
- can bound # creases (#disks) in terms of n
  & integral of “local feature size”
  (distance from x to another boundary point, dx)

**Open**: strongly polynomial bound possible
for any solution to fold & cut?
(conjecture not…)

**Simple fold & cut**: [Demaine, Demaine, Hawsley, Itô, Loh, Manber, Stephens 2010]
- all layers: (strongly) polynomial-time algorithm
  for polygons with margin
  - but # folds can be arb. large:
  - idea: guess a line of symmetry
  fold down to convex hull
  make “best possible” safe fold
  (reduce # vertices if possible)
  ⇒ graph gets smaller (smaller ≤ larger)
  - here use polygonness
  ⇒ convex hull (paper) gets smaller
- all layers: convex polygon ⇒ line of symmetry
- some layers: x-y-monotone orthogonal polygons
Flattening polyhedra: given polyhedral surface as piece of paper, can it fold flat at all? \[\text{[Demaine, Demaine, Lubiw 2000]}\] (without tearing)

**Connection to fold & cut:**

<table>
<thead>
<tr>
<th>2D fold &amp; cut</th>
<th>3D fold &amp; cut</th>
</tr>
</thead>
<tbody>
<tr>
<td>paper: 2D region</td>
<td>3D solid</td>
</tr>
<tr>
<td>cuts: 1D segments</td>
<td>2D polygons</td>
</tr>
<tr>
<td>fold: through 3D</td>
<td>through 4D</td>
</tr>
<tr>
<td>flat: down to 2D</td>
<td>down to 3D</td>
</tr>
<tr>
<td>so that: segments on line</td>
<td>polygons on plane</td>
</tr>
<tr>
<td>(\Rightarrow) flattening is boundary of</td>
<td>3D fold &amp; cut</td>
</tr>
</tbody>
</table>

**OPEN:**

- d-D fold & cut for \(d \geq 3\)? e.g. convex polyh.?  
- align all \(k\)-D faces, \(0 \leq k \leq d\), for \(d \geq 2\)?
- flattening based on 3D straight skeleton? \[\text{[Demaine, Lubiw 2000]}\]
  - possible for “thin convex prismatoids” \[\text{Demaine, Lubiw 2000}\]

**Flat folded state exists for orientable manifolds** \[\text{[Bern & Hayes 2008]}\]
- based on disk packing fold & cut \[\text{[see Ch. 18]}\]

**OPEN:**

- arbitrary polyhedral complexes
- continuous motion?
- connected configuration space of a polyhedral piece of paper?
  - no canonical state
  - not possible rigidly