2 meanings of "folding": (origami)
- folded state = description of paper after folding
- folding motion = continuum of folded states

we've focused on states, but in reality want motion

Equivalence: [Demaine, Demadoss, Mitchell, O'Rourke 2004]
any simple polygonal piece of paper has a folding motion into any desired folded state

Proof: (1D)

(2D)

OPEN: what if paper has holes?
unknotted polyhedral paper? (for flattening)
OPEN: do finite number of extra creases suffice, if target folded state does not touch itself?
  - above, all points become crease points

Rigid origami: what folds without extra creases?
  - faces of crease pattern = rigid polygons
  - creases = hinges

Example: [Balkcom, Demaine, Demaine, Ochsendorf, You 2006]
  paper shopping bag doesn't fold rigidly
  (for $h > w/2$, standard crease pattern)
  - 2 folded states: open & flat
  - no folding motions

Little known about rigid origami; looking for good open questions
**LINKAGES:**

Graph = vertices \( V \) & edges \( E \)  
(connectivity/combinatorial structure)

Linkage = graph + lengths of edges \( (l: E \to \mathbb{R}^d) \)  
(intrinsic geometry)

[+ coordinates for pinned vertices \( (p: V' \to \mathbb{R}^d) \)]

Configuration of a linkage in \( \mathbb{R}^d \)  
= coordinates for vertices \( (C: V \to \mathbb{R}^d) \)  
satisfying constraints of linkage  
\( \|C(v) - C(w)\| = l(v,w) \) for all \( \forall v, w \in E \);  
\( C(v) = p(v) \) for all \( v \in V' \)  
(allowing intersections for this lecture)

**Example:**

![Graph, linkage, two configurations]

Motion (of a linkage in \( \mathbb{R}^d \))  
= continuum of configurations \( (m: [0,1] \to C) \)
Configuration space = all configurations of a linkage
- view configuration of n-vertex linkage in $\mathbb{R}^d$ as (special) point in $\mathbb{R}^{dn}$:
  $C = (\ldots, \ldots, \ldots, \ldots, \ldots, \ldots, \ldots)$
  $d$ coords for $v_1$ $v_2$ $d$ coords for $v_n$
  $\Rightarrow$ configuration space = subspace of $\mathbb{R}^{dn}$
- motion = path/curve in configuration space
- square example: $n=4$, $d=2$
  $\Rightarrow$ configuration space lives in $\mathbb{R}^8$
  - 4 dimensions fixed by pinning
  - locally one dimensional; topologically:

![Configuration space diagram]

Degrees of freedom = local intrinsic dimension
of configuration space around configuration
- intuitively: $d \cdot (\# \text{unpinned vertices}) - (\# \text{edges})$
  (but in reality, some edges are extraneous — see L3)

Trajectory of a vertex in a linkage
= all points that vertex can reach in configurations
  (= projection of configuration space onto vertex’s coords)
Kempe's Universality Theorem: [Kempe 1876 had bug; Thurston; King 1999; Kapovich & Millson 2002; Abbott, Barton, Demaine 2008]

Any algebraic planar curve \( \Phi(x, y) = \sum_i c_i x^i y^{8i} = 0 \), intersected with any bounded disk, is exactly the trajectory of a vertex of some linkage.

Kempe’s “proof”:
- start with rhombus to constrain point \( p \) within disk:

- goal: constrain \( p = (x, y) \) to satisfy \( \Phi(x, y) = 0 \)

Main trick: use trig. to effectively “take logarithm”
- \( x = \frac{r}{2} \cos \alpha + \frac{r}{2} \cos \beta \)
- \( y = \frac{r}{2} \sin \alpha + \frac{r}{2} \sin \beta = \frac{r}{2} \cos (\alpha - \frac{\pi}{2}) + \frac{r}{2} \cos (\beta + \frac{\pi}{2}) \)
- apply trig. identity
  \( \cos A \cos B = \frac{1}{2} \left[ \cos (A+B) + \cos (A-B) \right] \)
  to polynomial \( \Phi(x, y) = \sum c_i x^i y^{8i} \)
  \( \Rightarrow \Phi(x, y) = c + \sum_i c_i \cos \left( r_i \alpha + s_i \beta + \delta_i \right) \)  
  \( \text{const.} \text{ const.} \text{ int.} \text{ int.} \text{ 0 or } \pm \frac{\pi}{2} \)
Kempe’s “proof” (cont’d)

- new goal: construct line segment of length $c_i$ and angle $r_i \alpha + s_i \beta + s_i$, for each $i$

- force final vertex on line $x = -c$ via large Peaucellier linkage

- build “machine” for angle arithmetic with ops:
  - multiply given angle by integer
  - add two given angles
  - copy an angle from one place to another
Kempe's gadgets:

**Contraparallelogram:**
- opposite sides equal & self-crossing (not parallelogram)
  ⇒ opposite angles equal; α determines β

**Multiplier:**
- k similar contraparallelograms sharing their β's ⇒ equal α's
- can be more efficient — (O(lg k)) edges — by repeated doubling, but this will not affect final complexity

**Additor:**
- use 2x multipliers to
  - bisect angle between segments
  - reflect x axis through bisector

**Translator:** two parallelograms
- opposite edges parallel & same length
  - make adjacent edges long (& same) for reach
- could use big rhombus — but this construction allows arbitrary length of input (or output) edge
Bug: [Kapovich & Millson]
- parallelograms can flip to contraparallelograms & vice versa via degenerate (flat) configuration
⇒ Kempe proved weaker result:
  trajectory includes desired poly curve & more
- fix for parallelogram:

  \[ \text{[Abbott & Barton 2004]} \]

- different, messier construction for complex polynomials
- fix for contraparallelogram:

Application:
Sign name via Weierstrass approximation theorem:
any continuous function \( f: [a,b] \rightarrow \mathbb{R} \) has an \( \varepsilon \)-approximate polynomial \( p \) such that \( |p(x) - f(x)| \leq \varepsilon \) for all \( x \in [a,b] \) for any \( \varepsilon > 0 \) (apply to each coordinate of curve)
Generalizations/strengthenings:
- curves/surfaces in d dimensions
- Θ(n^d) bars is optimal for degree n
- any compact semialgebraic set (d-dim)
  (bounded system of polynomial ≤ inequalities)
  as vertex trajectory
- coNP-hard to test rigidity
- configuration space = union of finitely many
  analytically isomorphic copies of any
  desired algebraic set (any # dim.)
  mapping & inverse have local power-series expansion

OPEN: what if edges are forbidden from crossing? [Shimamoto 2004]

PROJECT: implement Kempe applet

PROJECT: sculpture based on Kempe linkage/gadgets

PROJECT: design linkages for letters of alphabet
  (e.g. letter C: http://www.jimloy.com/cindy/cindy.htm)

Application: constructing algebraic numbers
  in origami via alignments
  [GFALOP 19.5: cf. Alperin & Lang 2006]