Infinitesimal rigidity: rigidity to the first order

Infinitesimal motion of a linkage configuration \( C \) = valid first derivative of a motion w.r.t. time, at time 0 = velocity vector \( d(v) \) for each vertex \( V \) preserving edge lengths to the first order:

\[
[C(v) - C(w)] \cdot [d(v) - d(w)] = 0.
\]

\( a \cdot b \rightarrow \text{dot product} \)

\[
a_x \cdot b_x + a_y \cdot b_y
\]

Rigidity matrix: "everything is linear, to the first order"

edge-length constraints form a linear system:

\[
\begin{pmatrix}
0 & 0 & C_x(v) - C_x(w) & C_y(v) - C_y(w) & 0 & 0 & C_x(w) - C_x(v) & C_y(w) - C_y(v) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
(d_x(v_1) \\
d_y(v_1) \\
\vdots \\
d_x(v_n) \\
d_y(v_n)
\end{pmatrix} = \emptyset
\]

\[\text{dn columns (d dim., n vertices)}\]

RIGIDITY MATRIX \( R \)

Infinitesimal motions = \( d \) for which \( R \cdot d = \emptyset \)

= kernel \( R \) = nullspace \( R \)

\( \leftrightarrow \text{linear subspace of some dimension: nullity } R \)

Infinitesimally rigid if nullity \( R = \frac{d(d+1)}{2} \leftrightarrow \text{rigid motions} \)
Rank-Nullity Theorem: rank $R + \text{nullity } R = \# \text{ cols. } = d \cdot n$

$\Rightarrow$ inf. rigid $\iff$ rank $R = d \cdot n - d(d+1)$ “full rank”

$\Rightarrow$ can test inf. rigidity in polynomial time using e.g. Gaussian elimination

**Generic point set** (more explicit definition than L9)

= all minors of rigidity matrix of complete graph
induced square submatrix on subset of rows & cols.
with nonzero determinant for some point set
(i.e. not identically zero, algebraically)
are nonzero for this point set

**Generic results:**
- almost every configuration is generic
- at generic configurations,
  rigidity = infinitesimal rigidity = generic rigidity [L9]
  $\Rightarrow$ randomized polynomial-time generic rigidity test:
  test infinitesimal rigidity of random realization
- if any realization is infinitesimally rigid
  then graph is generically rigid
  (else generically flexible with probability 1)

**Taking derivatives:** flexible $\Rightarrow$ infinitesimally flexible
i.e. infinitesimally rigid $\Rightarrow$ rigid

- but not vice versa:
Tensegrity = tens(ional inte)grity  

**PROJECT**: build tensegrity sculpture

- linkage but where each edge is either:
  - **bar** (as before): fixed length
  - **cable**: length can only decrease
  - **strut**: length can only increase

- configuration space becomes semi-algebraic polynomial inequalities

- motion, rigidity as before
- but not generic rigidity:

- infinitesimal motion (& rigidity):
  \[
  [C(v) - C(w)] \cdot [d(v) - d(w)] = \emptyset \quad \text{for bars } vw \\
  \leq \emptyset \quad \text{for cables } vw \\
  > \emptyset \quad \text{for struts } vw
  \]

\[\Rightarrow\] inf. motion space is a polyhedral cone
\[\Rightarrow\] inf. rigidity testable in polynomial time via linear programming
Equilibrium stress = real number for each edge $s: E \rightarrow \mathbb{R}$ such that $s(e) \geq 0$ for cables $e$ and $s(e) \leq 0$ for struts $e$. (push back in resistance)

**EQUILIBRIUM** $\sum_{w: \text{edge } vw} s(vw) \cdot [C(v) - C(w)] = 0$ for all vertices $v$.

- View $s(vw)$ as a scale factor on force along edge $vw$ felt equally by $v$ & $w$.
  - $s(vw) > 0 \Rightarrow$ push on $v$ & $w$ (resist compression)
  - $s(vw) < 0 \Rightarrow$ pull on $v$ & $w$ (resist expansion)
  - $s(vw) = 0 \Rightarrow$ no force

- Trivial equilibrium stress: $s(e) = 0$ for all $e$

**Example:**

equilibrium stress $| \text{force polygon} |$ no nontrivial equi. stress

**Duality in tensegrities:** [Roth & Whiteley 1981]

- Some equilibrium stress is nonzero on strut/cable $e$ $\iff$ every infinitesimal motion holds $e$'s length fixed
- Tensegrity is infinitesimally rigid $\iff$ every strut/cable is nonzero in some equilibrium stress & corresponding linkage is rigid
  - Replace cables & struts with bars
  - Proofs based on linear-programming duality
Spiderwebs: all-positive equilibrium stress (except on boundary) \(\Rightarrow\) infinitesimally rigid [Connelly 1982]

Origami tessellations:
- much history [Momotani 1984; Fujimoto 1982; Huffman 1960s, 1978; Resch 1968; Barreto 1997; Palmer 1997; Bateman 1990s; Verrill 1990s; Lang 2000s; Gjerde 2009]
- shrink-rotate algorithm:

  0) tessellate  1) shrink  2) rotate  3) fill  4) fold

- works for some shrink/rotate amounts \(\Leftrightarrow\) tessellation is a spiderweb [Lang & Bateman 2010]
Polyhedral lifting of a noncrossing configuration
\[ z \text{ coordinate for each vertex } z : V \rightarrow \text{IR} \]
such that each face remains planar
- assume outside face at \( z = 0 \) by rigid motion
- trivial lifting: \( z(v) = 0 \) for all \( v \)

Example:

Maxwell-Cremona Theorem: [Maxwell 1864; Cremona 1872]
one-to-one correspondence in noncrossing tensegrity between
equilibrium stresses & polyhedral liftings:
- negative stress \( \leftrightarrow \) valley edge
- positive stress \( \leftrightarrow \) mountain edge
- zero stress \( \leftrightarrow \) flat edge

**PROJECT**: implement program to illustrate
stress \( \leftrightarrow \) lifting correspondence
and/or stress \( \leftrightarrow \) inf. motion correspondence

**PROJECT**: virtual tensegrity building toy
- illustrate infinitesimal flexibility if any
Noncrossing linkages:
- configuration cannot have crossing edges
- configuration space smaller; still semi-algebraic

Locked linkage if configuration space is disconnected
i.e. no motion between some two configurations

Summary:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Trees</th>
<th>Chains</th>
</tr>
</thead>
<tbody>
<tr>
<td>2D</td>
<td>can lock</td>
<td>never locked</td>
</tr>
<tr>
<td>3D</td>
<td>can lock</td>
<td>can lock</td>
</tr>
<tr>
<td>4D⁺</td>
<td>never locked</td>
<td>never locked</td>
</tr>
</tbody>
</table>

Carpenter’s Rule Theorem: [Connelly, Demaine, Rote 2000/2003]
any linkage configuration of maximum degree 2
has a motion that
- straightens/convexifies all outermost open/closed chains
- is expansive: distance between any two vertices only increases

⇒ is noncrossing, by Δ inequality
Proof sketch of Carpenter's Rule Theorem:
- build tensegrity from linkage (edges \(\Rightarrow\) bars) + all possible struts (except where bar exists)
- linkage has expansive infinitesimal motion \(\iff\) tensegrity is infinitesimally flexible
\(\iff\) every equilibrium stress is zero (on struts)
\(\iff\) every polyhedral lifting is flat (on struts)
- detail: need to show stresses are equivalent in tensegrity vs. planarized tensegrity

\[ \begin{array}{c}
\bigtimes & \rightarrow & \bigtimes \\
& & \& \rightarrow \\
\end{array} \]

- here is where nested components get discarded
- slice hypothetical polyhedral lifting near max. \(z\):
  - peak case:
  - convex vertices \(\iff\) mountain edges
  - reflex vertices \(\iff\) valley edges
  - every polygon has \(\geq 3\) convex vertices
  \(\Rightarrow\) \(\geq 3\) incident mountains (positive stress)
  \(\Rightarrow\) \(\geq 3\) incident bars (no cables)
  - but max. degree 2
  - general case:
  - flat: need convex vertex to reach around reflex angle
  \(\Rightarrow\) flat except inside convex polygons
- integrate ordinary differential equation \(\Rightarrow\) expansive motion
6.849 Geometric Folding Algorithms: Linkages, Origami, Polyhedra
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