Locked & unlocked chains of planar shapes:

- simple locked examples:
  - 3 triangles
  - equilateral Δs
  - [M. Demaine 1998]

Adorned chain: view shapes as adornments attached to bar connecting hinges

- underlying chain linkage
- some flexibility in first & last shape

Adornment = shape + base

- shape = simply connected compact planar region
- base = line segment connecting 2 boundary pts.
- require base to be contained in shape

[Connelly, Demaine, Demaine, Fekete, Langerman, Mitchell, Ribo, Rote 2006–2010]
Slender adornment = walking along boundary from one base endpoint to the other monotonically increases distance to former (& decreases distance to latter)

= all inward normals hit the base 
  (for piecewise-differentiable shapes)
  = (possibly infinite) union of half-lenses: intersection of disks centered at base endpts. & halfplane on one side of base
  (⇒ can define slender hull = union of half-lenses thru every point in adornment)

Slender ⇒ not locked: expansive motion of underlying chain linkage preserves nonintersection of slender adornments

Intuition: consider two touching adornments
  - draw collinear inward normals from touching point \( x \)
  - resulting points \( a \) & \( b \) expand (vertices expand ⇒ points on bars expand)
  - two copies of \( x \) locally expand
  - in reality, this argument is tricky; can stay equal, to first order
  - possible with strict expansiveness [see SoCG 2006 proof]
Symmetric case: adornments reflectionally symmetric about their bases.

⇒ slender adornment = union of lenses

Stronger result for this case:

- instantaneous expansive motion of underlying chain linkage preserves nonintersection of slender adornments

Proof: take any two lenses of different adornments - nonintersecting before the motion.
- i.e. four disks have empty intersection

Kirszbraun's Theorem: [1934]

- if we instantaneously translate n disks with an empty n-way intersection according to an expansive motion on their centers, then they still have empty intersection.

(Annoying detail: Kirszbraun's disks include their boundary, but our disks might kiss - but Kirszbraun's Theorem holds for disks excluding their boundaries, by taking limits of slightly smaller disks)

If initially intersecting, can show area of union only increases [Bezdek & Connelly 2002 ~Kneser-Poulsen conj.]
Proof that slender $\Rightarrow$ not locked: (general case)

- not true for instantaneous: $\bigcirc \Rightarrow \bigcirc$

- take any 2 half-lenses of different adornments
- consider transition time from not intersecting to intersecting $\Rightarrow$ touching

- 3 types of touching:
  1. bases of both
     - nonintersection guaranteed by underlying chain linkage
  2. base of one
     - can add symmetric lens of other, & just consider base of first (X)
     - no intersection by symmetric case
  3. base of neither
     - can add symmetric lens of both
     - again no intersection by symmetric case \(\square\)

Carpenter's rule theorem $\Rightarrow$ straighten/convexify any slender-adorned (non-self-touching) chain $\Rightarrow$ connected config. space of open chains

- not true of closed chains:
**OPEN:** which adornments never lock in a chain? (like slender)

**Triangles:** not locked if angles opposite base $\geq 90^\circ$ (right or obtuse)
- locked (nearly) identical equilateral triangles:

  - can stretch/shrink in y coord. to make locked example with any angle $< 90^\circ$

**Proof:** show self-touching version rigid
  $\Rightarrow$ strongly locked  [Lecture 11]

- Rule 1
- Rule 2

**Lemma:** any motion of a convex polygon decreases $\geq$ two angle $\Rightarrow$ rigid
Locked triangles proof: (cont'd)

- Clearly rigid if zero-length struts were bars
- Set \( s(AB) = -s(AB') < 0 \) \( \Rightarrow \) A in equilibrium
- Set \( s(BC) = s(AB) = -s(B'C') = -s(AB') < 0 \)
  \( \Rightarrow \) force on B, B' vertical \( \Rightarrow \) in equilibrium if
  set \( s(B, AB') = s(B, B'C') < 0 \) appropriately
- Set \( s(C'D'), s(D'DC), s(D'DE) < 0 \) unique up to scale
to put D' in equilibrium; scale very small
- Set \( s(CD) = -s(C'D') \) \( \Rightarrow \) D in equilibrium (inverse of D')
- \( s(BC) < 0 \) dominates \( s(CD) \) \( \Rightarrow \) can set \( s(C, C'B') \) &
  \( s(C, C'D') < 0 \) to put C & hence C' in equilibrium \( \square \)

\[ \text{OPEN: locked chain of unit squares?} \]
Hinged dissections: [Abbott, Abel, Charlton, Demaine, Demaine, Kominers 2008/2010]

there's an open chain of hinged polygons that folds continuously into any desired finite set of polygons of equal area (without collision)

Idea:
1. start with any dissection ~ no hinges set of polygons that can be assembled into each target polygon
2. hinge arbitrarily (or to match one target)
3. subdivide pieces & add hinges to enable each desired assembly
4. subdivide to make pieces slender \(\Rightarrow\) motion

(1): [Lowry 1814; Wallace 1831; Bolyai 1833; Gerwin 1833]
- cut each polygon into triangles
- cut each triangle into rectangle
- dissect each rectangle into rectangle of height \(\varepsilon\) [Montucla 1778]
  (this is the hard step ~ skipped here)
- string rectangles from one polygon into one long height-\(\varepsilon\) rectangle
- overlay these cut patterns of \(4/3 \times 3\) rect...
3: maintain tree hinging
   - key step: effectively move rooted subtree to attach at any other vertex

   - 2 of these ops. brings two vertices together
   - repeat until vertices together for target
   - repeat for each target polygon

# pieces: without care, can roughly double for each step
   - can be improved with care

4: triangulate pieces
   - cut each \( \triangle \) at centroid
     \( \Rightarrow \) obtuse
   - connect into chain by "walking along the outside of the tree (Euler tour)"
     \( \Rightarrow \) slender chain
     \( \Rightarrow \) not locked