Recall:

<table>
<thead>
<tr>
<th>convex polyhedra</th>
<th>edge unfolding</th>
<th>general unfolding</th>
</tr>
</thead>
<tbody>
<tr>
<td>OPEN</td>
<td>ALWAYS</td>
<td>OPEN</td>
</tr>
<tr>
<td>nonconvex polyhedra</td>
<td>NOT ALWAYS</td>
<td></td>
</tr>
</tbody>
</table>

Vertex unfolding: [Demaine, Eppstein, Erickson, Hart, O'Rourke 2003]
- different relaxation of edge unfolding
- still cut only along polyhedron edges (all!)
- require connectivity just through vertices (vs. edges) (like hinged dissection)

Example:

Universality: every connected triangulated manifold has a vertex unfolding, computable in linear time

1. construct facet path going from facet to facet along vertex adjacencies, visiting every facet exactly once, not repeating a vertex twice in a row
   [next page; also Bartholdi & Goldsman 2004]
   \( \text{but } O(n^2) \text{ time} \)

2. unfold each facet into vertical slab:

\[ \Rightarrow \text{no overlap} \]
Vertex unfolding: (cont'd)

Constructing a facet path for 2D surfaces:

1. Cut edges until facets are connected in tree-like fashion
   - only removes connections
   - only harder to find path
   - duplicates vertices but can repeat
   - triangulated “polygon” (may self-overlap)

2. Color “ears” = Δs with one adjacent triangle & two boundary edges

3. Color cars in what remains

4. Remove second ear & 1 or 2 first ear(s):
   - either “dunce cap” or “Mickey Mouse”
   - recurse on remainder
   - put back with or
   - base case: nothing or
   - every vertex but 2 from base case have even number of connections

5. Connect components by local switches:

6. (Noncrossing) Eulerian path
   - bonus: get cycle ⇔ not “checkered”:
     2-color Δs, only red Δs on boundary
Vertex unfolding: (cont’d)

General vertex unfolding is trivial: triangulate

**OPEN**: does every convex polyhedron have a vertex unfolding?

- facet path $\Rightarrow$ vertex unfolding:
  - no slabs \[\text{repeat 3 times}\]

- facet path may not exist:
  - truncated cube has
    - 8 disjoint $\Delta$s & 6 $\bigcirc$s
  - not enough $\bigcirc$s to put between $\Delta$s

**OPEN**: does every polyhedron with holeless faces have a vertex unfolding?

\[\text{cf.}\]

\[\text{diagram}\]
Orthogonal polyhedra: (all faces perpendicular to coord. axis) generally unfoldable if genus $\neq 0$

[Damian, Flatland, O’Rourke 2007]

Proof: slice through every vertex with y-plane (xz)
- y-faces encompassed = side faces
- $x \& z$-faces (yz & xy) = band faces
- band faces between two y-planes form a cycle = band
- bands form a tree structure, connecting parent to child with thin side faces
- idea: visit bands in roughly depth-first order
  - unfold to proceed always rightward
  - side faces just attach above/below
- unfold leaf band with spiral path:

- visit $y$ children with alternating path:

- similarly visit $y$ children
- double-back along entire subtree to return to parent $t$
Orthogonal polyhedra: (cont’d)

Grid = slice through every vertex with x,y,z-planes
Grid unfolding = just cut along grid

OPEN: does every orthogonal polyhedron have a grid unfolding?
- analog of edge unfolding

Refinement = divide each grid rectangle into $k \times k$

Summary:

- general
- general vertex-unif.
- Manhattan towers
  - connected y=0 base, y-plane slices shrink as y increases
- orthoterrain (rect. base)
- orthostacks
  - every y-plane slice is connected
- orthostacks vertex-unif.
- orthoconvex orthostacks
  - y-plane slices are orthogonally convex: x & y slices connected
- orthotubes
  - unit cubes connected face-to-face in open/closed chain
- well-separated orthotrees
  - unit cubes connected face-to-face in a tree

$\Rightarrow$ no two branching cubes are adjacent

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<td>[DFO 2007]</td>
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<tr>
<td>Orthoterrain (rect. base)</td>
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Orthogonal polyhedra: (cont'd)

OPEN: \( n^{O(1) \times n^{O(1)}} \) refinement in general?

OPEN: \( \Omega(n) \times \Omega(n) \) lower bound?

OPEN: power of \( O(1) \times O(1) \) refinement?
  - e.g. orthotrees

OPEN: power of 1x1 refinement?
  - e.g. orthostacks, Manhattan towers

OPEN: orthogonal polyhedra of higher genus?

OPEN: general unfolding of arbitrary polyhedra of genus 0?

PROJECT: implement \( 2O(n) \times 2O(n) \) method for unfolding orthogonal polyhedra of genus 0
Cauchy's Rigidity Theorem: [Cauchy 1813; Steinitz 1934]
if two convex polyhedra are combinatorially equivalent (same graph/incidence structure) & corresp. faces are congruent, then polyhedra are congruent

Proof by contradiction: consider counterexample P, P'
- for each vertex pair v, v': slice polyhedron with ε-sphere at v, v'
  ⇒ spherical polygon, edge lengths = face angles, angles = dihedral angles
- faces congruent ⇒ edge lengths match in V vs. V'
- P, P' incongruent ⇒ angles don't match for some v, v'
- label vertex in V's spherical polygon + if larger angle in V, ∅ if equal angles, and - if smaller angle in V
- can't be all + (& ∅) or all - (& ∅) by: Cauchy Arm Lemma: opening all angles of a convex open chain increases distance of endpts. (in plane & sphere)
  ⇒ edge length would not be preserved
  ⇒ ≥ 2 alternations ++--+-++--+-++--(++++)
  - can't be just 2 by Cauchy Arm Lemma:
  ⇒ ≥ 4 alternations +---+++---
⇒ # alternations ≥ 4V in subgraph of +/- edges
Proof of Cauchy's Rigidity Theorem: (cont'd)
- +/- labels extend to entire edge of polyhedron
- +/- alternation at vertex corresponds one-to-one
- +/- alternation in incident face
- \( \leq 2k \) alternations in face of \( 2k \) or \( 2k+1 \) edges
  \( \Rightarrow 4V \leq \# \text{alternations} \leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + 7f_7 + \cdots \)
- \( E = \frac{1}{2} (3f_3 + 4f_4 + 5f_5 + \cdots) \) (handshaking)
- \( V = 2 + E - F \) Euler's Theorem
  \( = 2 + \frac{1}{2} (f_3 + 2f_4 + 3f_5 + \cdots) \)
  \( \Rightarrow 4V = 8 + 2f_3 + 4f_4 + 6f_5 + \cdots \Rightarrow \text{contradiction} (+8) \)

Uniqueness of folding: [Alexandrov 1941]
- suppose you glue polygon's boundary to itself
- what convex polyhedra can it form?
- every edge will be a shortest path
- draw all shortest paths between vertices
  (points of nonzero curvature)
  \( \Rightarrow \) fix combinatorial & facial structure
- Cauchy's Rigidity Theorem \( \Rightarrow \leq 1 \) convex realization

NEXT LECTURE: when is there \( > 1 \) convex realization?